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Games in Energy Markets

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by

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Games in Energy Markets

Xuwei Yang

To my parents Qinghua Yang and Lirong Yuan.

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Curriculum Vitæ

Xuwei Yang

Education

University of California, Santa Barbara 2010-2017
Ph.D. Statistics and Applied Probability with Emphasis in Financial Mathematics & Statistics
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Research Interest

Applied Probability, Stochastic Processes, Financial Mathematics, Optimal Stochastic Control, Stochastic differential Games, Mean Field Games.

Academic Positions

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Postdoctoral Scholar at Department of Mathematical Sciences.

Publications

1. Mean Field Games Approach to Production and Exploration of Exhaustible Commodities, with M. Ludkovski, arXiv preprint (2017), arXiv:1710.05131.
2. Dynamic Cournot Models for Production of Exhaustible Commodities under Stochastic Demand, with M. Ludkovski, *Commodities, Energy and Environmental Finance*, R. Aid, M. Ludkovski, and R. Sircar, eds., Fields Institute Communications, AMS, 2015.
3. *Financial Engineering and Risk Management Techniques(Chinese Edition)*, translator of chapters 3, 6, 7, 8, 9, 11, with Lixin Liu et al., China Machine Press, 2009. ISBN9787111260905.

Invited Talks and Presentations

1. *Mean Field Game Approach to Production and Exploration of Exhaustible Commodities.*
   Special Session on Financial Mathematics,
   American Mathematical Society Sectional Meeting, Brunswick, Maine, September, 2016.
2. *Mean Field Game in Energy Markets.*
   Minisymposium on Mean Field Games,
   The International Conference on Industrial and Applied Mathematics, Beijing, China, August, 2015.

3. *Mean Field Game Approach to Production and Exploration of Exhasutible Commodities.*
Special Session on Recent Advances in Mathematical Modeling of the Financial Markets,
American Mathematical Society Sectional Meeting, East Lansing, Michigan, March, 2015.

4. *Exploration and Production of Exhaustible Commodities.*
SIAM Minisymposium on Recent Advances in Financial Mathematics,
Joint Mathematics Meetings, Baltimore, Maryland, January 2014.

5. *Dynamic Cournot Models for Production of Exhaustible Commodities under Stochastic Demand.*
Workshop on Stochastic Games, Equilibrium, and Applications to Energy and Commodities Markets,
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1. Invited participant June, 2015
Financial Mathematics Workshop, Mathematics Research Communities program, American Mathematics Society, Salt Lake City, UT, U.S.

2. Visiting Graduate Researcher (Receive a funding of \$6000) March-June, 2015
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3. Invited Visitor (Received a funding of \$1000) July, 2014
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Pacific Institute of Mathematical Sciences, University of British Columbia, Vancouver, BC, Canada.

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Teaching Experiences

1.Instructor (Worcester Polytechnic Institute) 2015-2017

- •MA528 Measure-Theoretic Probability Theory
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2.Teaching Associate (University of California, Santa Barbara) Summers 2012-2014

- •PSTAT109 Statistics for Economics
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Honors and Awards

1.Society of Industrial and Applied Mathematics Travel Funding 2015

2.American Mathematical Society(AMS) Student Travel Support 2015

3.Doctoral Students Traveling Grant, University of California, Santa Barbara 2014

4.Teaching Assistantship, University of California, Santa Barbara 2010-2015

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Abstract

Games in Energy Markets

Xuwei Yang

We study energy markets in game theoretic framework. The energy markets consist of two types of energy producers: exhaustible producer and renewable producer. An exhaustible producer produces energy with exhaustible resources, such as oil. The resource reserves of each exhaustible producer diminish due to production, and also get replenished with costly effort to explore for new resources. This exploration activity is modeled through a controlled point process that leads to stochastic increments to reserves level. A renewable producer uses renewable resources, such as solar power, to produce energy. The renewable resources are infinite, but costly in production. Each producer chooses optimal controls of production quantity and exploration effort (exhaustible producers only), in order to maximize individual profit that equals his quantity of production multiplied by market price, minus costs of production and exploration. The producers interact with each other through the energy price that is a function of aggregate production, as one's profit does not only depend on his own production quantity, but also depends on the total quantity of all other producers. We aim to study the equilibrium total production and price.

In Chapter 2 we study the game between an exhaustible producer and a renewable producer under stochastic demand that switches between different regimes. We study how the regime changes and the relative cost of production, which is a proxy for market competitiveness, affect game equilibria, and compare with the case of deterministic demand. A novel feature driven by stochasticity of demand is that production may shut down during low demand to conserve reserves.

In Chapter 3 we study game with a continuum of homogeneous exhaustible producers. Mean field game approach is employed to solve for an approximate Markov Nash equilibrium of the game. We develop numerical schemes to solve the resulting system of partial differential equations: a backward Hamilton-Jacobi-Bellman (HJB) equation for the game value function of a representative producer and a forward transport equation for the distribution of the reserves levels among all producers.

In Chapter 4 we study a time-stationary mean field game model, in which the reserves level remains invariant due to the counteracting effects of production and exploration. We also study the impact of uncertainty in the regime that the exploration process becomes asymptotically deterministic, so that discovery of new resources happens at high frequency with small amount of each discovery.

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Chapter 1

Introduction

We study production and exploration of exhaustible resources for the purpose of generating energy. Exhaustible resources, such as oil, coal, and ores have great importance in the functioning of the whole economic system. Dwindling oil reserves and the resulting impact on energy supply and price is of fundamental importance to the functioning of the whole economic system. Energy production with the resources generates revenue but lowers remaining reserves for further production. Exploration for new resources will likely lead to discoveries that add to reserves for production, though exploration is costly and discoveries occur in an uncertain way. In oil industry, for instance, exploration activities include research and development of new drilling techniques, and putting human labors and facilities into scanning geographic areas for new resources.

Exploration is one of the main interests in the research. Mathematically we use a point process to model the exploration. Jumps of the point process mark the discovery times. The amount of each discovery can be random in general, but we assume it is a constant positive quantity without loss of any generality. The intensity of the point process is subject to the control of producers. More effort input leads to higher frequency of new discoveries as well as higher exploration costs.

Energy markets can be viewed as companies competing with each other in energy production and exploration for new resources, for the purpose of profit maximization. Each energy company is regarded as a producer. It is of great interest to study the competition between the energy producers and the resulting equilibrium prices and total supplies. Due to the competitive nature of energy markets, game theory is a useful tool to study the outcome of the competition. Game theory deals with strategical interactions among multiple decision makers, who are also called *game players* in game theoretic language. Each player has an objective function and chooses strategic variables to optimize the objective function. The strategic variables are also called *controls* from the perspective of optimization. The players interact with the others in the way that each one's objective function involves the strategic variables of the others, thus one has to make decision by taking into account the strategies of the others. Producers in energy markets can be viewed as players competing with each other in order to maximize profit. The strategic variables of players in the research are production quantity and exploration effort. We work in a *Cournot game* framework in which players choose quantities of energy production and receive profit based on a single market price determined through aggregate supplies. In contrast to the Cournot game, where the strategic variable is quantity of production, the other type is *Bertrand* game, in which price is the strategic variable of the players. Harris, Howison, and Sircar [31] studied Cournot model of exhaustible resources, while Ledvina and Sircar [39] studied Bertrand model. The game we study is non-cooperative, since producers make decisions independently without any cooperation.

The players competing with each other by choosing strategies of production and exploration, in order to maximize expected profit. The game equilibrium of our interest is *Nash equilibrium*, which is a set of all players' strategies such that no one can better off by unilaterally changing individual strategy. A single player's decision depends on the reserves levels of all players, thus the controls take the feedback form. Moreover, the control variables are Markovian, that is, the decisions are based on current reserves level that contains sufficient information of the past. Since the players choose strategies from the admissible set of Markov feedback controls, the game equilibrium we study is Markov feedback Nash equilibrium.

Due to the game nature and stochasticity, the research is embedded in the framework of *stochastic differential game*. Stochastic differential game is closely related to *stochastic control theory*. Isaacs [35] is a good reference for introductory differential game. We mention [24] as a reference for stochastic control and stochastic differential game. The basic formulation of stochastic control involves a dynamic system driven by stochastic factors, whose state evolution can be influenced by exercising controls. Associated with the system is an objective function which depends on the state process and controls over either a finite or an infinite time horizon. The main goal is to find the control that can achieve the optimal (minimal or maximal) value of the objective function. The objective function under the optimal control is called value function. *Dynamic programming* method is employed to solve for optimal control, which equates the value function under optimal control to the value realized in a local infinitesimal time interval under optimal control plus the value function after that. By dynamic programming method, we obtain a partial differential equation of the value function in terms of time and state variables, which is formally called *Hamilton-Jacobi-Bellman* (HJB) equation. Optimal control is linked to the value function through the HJB equation. thus we can solve for optimal control by solving the equation. In stochastic differential games there are players associated with a stochastic dynamic system, on which they can exercise controls. Each player has an objective function depending on the state variable of the stochastic dynamic system and the controls. All the players are interrelated in such a way that each player's value function involves other players' state variable and controls as well as his own. The controls in the game theoretic context are also called *strategies*. Each player has to consider all the other players' strategies while making his own strategy. Since the objective function value each player achieves is though game, we call it *game value function*. We use dynamic programming method to solve for the equilibrium strategies. Since each player's game value function involves other players' controls, we need to freeze other players' controls when we derive the partial differential equation for one player. The system of partial differential equations of game value functions is called *Hamilton-Jacobi-Bellman-Isaacs* (HJB-I) equations. HJB-I equation is a main tool we use to solve for Nash equilibrium strategies in the research.

The research is organized as follows. In Chapter 2, we study Cournot game between two producers of different resource types under stochastic demand. Particularly we study how stochastic demand affects the Nash equilibria of the game. In Chapter 3, we employ mean field game approach to find approximate Markov Nash equilibrium of Cournot game with a continuum of exhaustible producers. Numerical schemes are developed to solve the resulting system of partial differential equations. In Chapter 4 we study a time-stationary mean field game model, in which the reserves level remains invariant due to the counteracting effects of production and exploration. We also study the effect of randomness of exploration process on equilibrium production and reserves distribution.

In Chapter 2 we study a game model between two energy producers of different resources: producer 1 that extracts an exhaustible resource (oil) and has to worry about diminishing reserves; and producer 2 that extracts a renewable resource (green energy such as solar power) and therefore has infinite reserves. Two stochastic state variables are considered: current reserves level of the exhaustible player 1 and demand level. Moreover, players have a total of three controls, namely productions rates for players 1 and 2, as well as exploration effort for producer 1. The exhaustible reserves level follows piecewise deterministic trajectories, smoothly decreasing due to production and experiencing constant-size jumps upon new reserves discovery. These upward jumps of fixed size mimic discrete discoveries of new oil fields or new oil recovery technologies that take place abruptly. The demand level is modeled as a continuous-time Markov chain that switches among different regimes to mimic business cycle fluctuations.

The two producers in Chapter 2 compete through a Cournot framework, in which producers choose quantity of production and receive profit based on a single market price determined through aggregate supply. Costs of the exhaustible player are driven solely by the costly (convex) exploration effort; her production costs are taken to be zero. On the contrary, the green producer has a positive marginal cost of production but inexhaustible resources. These production costs of player 2 are a proxy for the amount of competition. They also reflect the present reality of non-renewable energy production as being the cheaper incumbent against the new renewable entrants. The Cournot framework is used because at the macro level, energy is perfectly substitutable and so the two producers' products are in direct competition. Producers' profits are equal to the quantities of production multiplied by the market price, minus the cost of exploration (player 1) or production (player 2). The game value functions of the two players are the discounted cumulative expected profits starting with certain initial reserves level and demand regimes.

The aim of Chapter 2 is to study the game between the exhaustible producer and the green producer in terms of dynamic Nash equilibria, and particularly the impact of stochastic demand on the game equilibria. The model is cast in continuous-time so as to allow use of the Hamilton-Jacobi-Bellman-Isaacs methodology that reduces computational analysis to study of coupled systems of differential equations. We use dynamic programming method to obtain a system of HJB-I equations of the players' game value functions. Those equations are first-order nonlinear forward-delayed ordinary differential equations. The equations are first-order due to the lack of diffusive stochastic factors such as Brownian motion. The forward-delay term is due to the controlled point process that marks the moment of new discovery. Moreover since only a single agent has reserves, there is just one continuous state-variable, effectively allowing us to deal only with ordinary differential equations rather than partial differential equations. Particularly the equilibrium production of the two players are piecewise-linear in the derivative of game functions, which leads to piecewise-defined with free boundaries between adjacent pieces. Towards the end of chapter 2 we also mention some extension of the Cournot model by considering stochastic cost of renewable energy production.

In chapters 3 and 4, we study Cournot game with a continuum of homogeneous exhaustible producers. Each producer has production quantity and exploration effort as control variables. The single market price is determined by all producers' total production. Both production and exploration are assumed to be costly. Each player has game value function that equals to the discounted amount of profit minus costs, where the profit and costs are realized under the equilibrium strategies of all players. We aim to study equilibrium production quantities and reserves distribution in energy markets with a large population of competing producers. Particularly, we want to understand how exploration activities affect the long-term market organization, and how the exploration uncertainty permeates the solution.

According to the classical HJB-I method for an N-player game, each player is associated with an HJB-I equaiton and thus the model involves a system of N

partial differential equations which is intractable. We employ mean field game approach to model energy markets with a continuum of exhaustible producers. The mean field game model reduces the system of N partial differential equations to a system of two doubly coupled partial differential equations: one is the HJB equation of a representative player's game value function; and the other is the transport equation of all players' reserve distribution. The market price, directly related to the total production of all the producers, enters the game value function of the representative producer as the mean field term. The representative player chooses his optimal quantity of production and exploration effort depending on all the other players strategies through the mean field term market price.

1.1 Games with Exhaustible Resources

It is worthwhile mentioning models of single-agent before considering games with more than one agents. There is a long literature on optimal economic behavior of a natural resource monopolist extracting non-renewable resources. Hotelling [32] found that without discovery exhaustible resource price grows at inter-temporal discount rate. Pindyck [44] studied a deterministic model of exploration for exhaustible resources. In the model exploration was assumed to be incremental and represented as a deterministic reserve addition. It was shown that the resulting resource shadow price, corresponding to the marginal value of additional reserves will firstly decrease and then increase as reserves run low. As extensions to Pindyck [44] there is a series of works studying exploration. Arrow et al. [2, 21, 30, 47] represented exploration, which is punctuated by large discoveries, as a point process. Pindyck [45] further studied a model in which the total size of reserves is unknown. The dynamics was then described via a stochastic differential equation with controlled volatility and drift. Dasgupta and Heal [19] is a comprehensive reference for literature on exhaustible resources up through the 1970s.

The first paper that rigorously treated a dynamic non-cooperative model for exhaustible resource extraction was published by Harris, Howison, and Sircar [31]. They studied N-player continuous-time Cournot game in which firms choose production quantities. The games were characterized by a system of nonlinear HJB partial differential equations which were analytically and numerically hard to resolve. They also analyzed the problem when there is an alternative, but expensive, technology (for example solar power for energy production). They illustrated the two-player problem by numerical solutions, and discussed the impact of limited oil reserves on production and oil prices in the case of two-player model.

Ludkovski and Sircar [40] studied a related model that allowed for stochastic evolution of reserves by considering exploration that can lead to discovery of new reserves. This analysis was motivated by the oil market where E&P (exploration and production) efforts total many billions of dollars a year. In that sense, while oil is exhaustible, it is also replenishable since there is a difference between total abstract reserves on Earth, and what is actually commercially "proven" and drives production decisions. With exploration, players have two complementary choices regarding running down existing reserves and expending effort in the hopes of finding new reserves. In particular, players may never fully "leave" the game since they can periodically resurrect themselves by ongoing discoveries. In [40], they firstly treated the case of a monopolist who produces and may undertake costly exploration to replenish his diminishing reserves. Then a stochastic game between an exhaustible producer and a "green" producer was studied. The new discoveries were modeled through a controlled jump process with intensity given by exploration efforts. The game between the two players led to a study of systems of non-linear first-order delay ordinary differential equations with implicit boundary conditions. The delay term and implicit boundary conditions were due to the nature of jumps in the model.

In my work [41] *Dynamic Cournot Models for Production of Exhaustible Commodities under Stochastic Demand*, which is the main content of Chapter 2, I extended the work [40] by studying the effects of stochastic demand on the equilibrium of the dynamic Cournot game between an exhaustible resources producer and a renewable resources producer. The state variable is the reserves level of exhaustible resources, which decreases at a (controlled) production rate and increases through a random process that has discrete increment at a controlled rate (this process is formally called *controlled point process*). The market price is a negative function of the aggregate production of the two producers. The game function of exhaustible producer is the total discounted profit over infinite time horizon that is determined by the total revenue(product of price and production) minus the exploration cost. The game function of renewable producer is the total revenue minus production cost. The game functions of the two players are coupled through the market price which is a negative function of their total quantity of production.

We considered stochastic demand as a simulation of macroeconomic volatility. The exogenous stochastic demand factor is modeled through a continuous-time Markov chain that switches between high and low regimes(it can be generalized to a process such as an Itô diffusion, but two-regime setting is sufficient to represent the market demand ups-and-downs volatility). The Markov chain enters the linear price function as a coefficient that moves the price between high and low regimes. We studied how the demand regime changes affect game equilibria, and compared with the case of deterministic demand in [40].

Due to demand regimes switching, each player is associated with one game function in each regime, which leads to a total of four game functions for two players in two regimes. In infinite time horizon, the model is time stationary. Thus the HJB-I partial differential equations become HJB-I ordinary differential equation, with time-derivative term vanishing. Due to the random discrete increment term in the dynamics of the state variable, the HJB-I equations involve forward-delay terms and implicit boundary conditions. The optimal control of the two players depends on the value function of the exhaustible producer, thus the HJB-I ordinary differential equations of the exhaustible producer is autonomous of those of the renewable producer. It is sufficient to analyze the HJB-I ordinary differential of the exhaustible producer to obtain the equilibrium production strategies of the two players.

The major challenge is that the function forms of the HJB-I ordinary differential equations are piecewise-defined with free boundaries between each two adjacent pieces. The free boundaries occur because the optimal controls of the two players are piecewise linear in the derivative of the game function of exhaustible producer. The interaction of the forward-delay term and the piecewise-defined functional form of ordinary differential equation poses a challenge. To the best of our knowledge, there is no reference about well-posedness of such equations. But using numeric method we solve an approximation to the system of equations that is guaranteed to be well-posed. To deal with the forward-delay term in the equations, we use an iterative scheme that starts without the delay term, and iteration goes on by taking the data from the last iteration to substitute the forward-delay term. In each iteration, we use fourth-order Runge-Kutta scheme to solve for the system of ordinary differential equations. This iterative scheme was proved to be convergent both analytically and numerically, according to [40].

Due to stochastic demand, the game equilibria become more complicated than the deterministic demand model in Ludkovski and Sircar [40]. A novel finding in the research is a new possible game equilibrium due to the stochastic demand. In the low regime, it is possible that the exhaustible production shuts down thus the renewable producer monopolizes the market. Exhaustible production shuts down for two reasons: one reason is the difference between high and low demand regimes is large thus it is profitable to shut down production in low regime in order to save reserves for production in high regime; the other reason is that the average holding time in low regime is short enough thus extra profit made in high regime can compensate the loss in low regime due to production shutdown. Production shutdown also leads to extra mathematical difficulty, because in this situation the derivative of game function zeros out, the ordinary differential equation in the low regime degenerates into an algebraic equation. In computation we have to detect when this happens, and then we need to switch to a new ordinary differential equation derived from the algebraic equation.

Since the stochastic demand is a simulation of macroeconomic volatility in the research, it is interesting to study how the game equilibrium changes as the frequency of regime switching changes. We use the asymptotic expansion of the game functions in terms of the average holding times in the two regimes, and by letting the average holding times in both the two regimes goes to zero we obtain the ordinary differential equation for limiting game function, which is again a piecewise-defined ordinary differential equation with forward-delay term and implicit boundary condition. By numerically solving the ordinary differential equation we obtain the equilibrium production and exploration when market demand volatility is high.

The single-agent and two-player models [40, 41] will be extended to a model with a continuum of players [42], which is introduced in Chapters 3 and 4. We employ mean field game approach to find approximate Nash equilibrium of the game, which will be introduced in the following Section 1.2.

1.2 Mean Field Game Approach

The second part of the research studies energy market with a continuum of producers. Mean field games (MFG) approach is applied to model the strategic interactions among a continuum of players and the resulting equilibrium of production, exploration, and distribution of resources reserves. In a differential game model with a finite number N of players, their equilibrium strategies can be determined by a system of Hamilton-Jacobi-Bellman-Isaacs (HJB-I) equations derived from the dynamic programming principle. The dimension of the system in general increases as the number N of players increases, which makes the game model intractable for large N. Mean field game (MFG) approach simplifies the modeling by considering equilibria with a continuum of homogenous players; the respective finite dimensional game state translated into a measure η. The main idea is to consider an optimization problem of the representative agent; the latter becomes a regular stochastic control problem with the competitive effect captured via a certain aggregate mean-field interaction driven by η. In turn, the aggregate behavior of the players implies dynamics on the distribution η of agent states. This leads to a system of two infinite-dimensional partial differential equations (PDEs) which is viewed as an approximation to the N-system of finite-dimensional PDEs in the original finite-N setup.

The MFG framework was introduced by Lasry and Lions [36, 37, 38] and Caines, Huang, and Malhame [34]. A formal introduction to the basics of mean field games and mean field type control can be found in Bensoussan, Frehse, and Yam [3]. Cardaliaguet [6] studied a system of first order mean field game equations with local coupling in the deterministic limit. A system of (possibly degenerate) second order mean field game partial differential equations was analyzed in [7], and existence and uniqueness of suitably defined weak solutions were proved. Carmona and Delarue [13] provided a probabilistic analysis of a large class of stochastic differential games for which the interaction between the players is of mean-field type, and proved that solution of mean field game indeed provides approximate Nash equilibria for games with a large number of players. Bensoussan, Sun, Yam, and Yung [4] provided a comprehensive study of a general class of linear-quadratic mean field games. Mean field games between a major player and a continuum of minor player can be found in [33, 43, 15]. Mean field game models with local coupling can be found in [5, 6, 7]. Cardaliaguet, Lasry, Lions, and Porretta [8] studied a locally coupled mean field game model defined on a finite time horizon and showed that the system converges to a stationary mean field game as time horizon tends to infinity. It was studied in [9] that a mean field game model with nonlocal coupling on a finite time horizon converges exponentially to the associated stationary mean field game model, a time horizon tends to infinity.

Achdou, Camilli, and Capuzzo-Dolcetta [1] introduced finite difference schemes for stationary and evolutive mean field game models, and proved convergence of the numerical schemes under various assumptions on the coupling operator. Carlini [10, 11] proposed fully discrete semi-Lagrange schemes for mean field game systems of first order and second order, and proved convergence of the numerical schemes. In [29, 16, 17], iterative schemes were used to solve the coupled mean field game partial differential equations associated with dynamic Cournot models for production of exhaustible resources.

The mathematical significance of the MFG approach is that it provides a tool to study (differential) games with a large population of players. Specifically, the MFG setup leads to an HJB equation to model a representative player's strategy, and a transport equation to model the evolution of the distribution of all the players' states. In our context, the states are the reserves' levels, and the interaction is via the market price p that is related to total production across all the producers. Thus, p enters the game value function of the representative producer as the mean field term and drives his optimal quantity of production and exploration efforts conditional on the other producers' strategies. In turn, the distribution of reserves is driven by the latter production rates and exploration efforts.

The study of exhaustible resources oligopolies using MFGs was initiated in Guéant et al. [28, 29] who studied a mean field Cournot model with a linear-quadratic production cost function, and a stochastically fluctuating reserves process. They further introduced generalized models that consider value function depending on further factors, e.g. they included a ranking effect into the value function. They also considered competition between producers of two types of resources. Graber [25] introduced a linear quadratic mean field game model of exhaustible resource production, in which the reserves process of a representative player is influenced by a common noise with all other players, as well as its individual noise. [25] at the beginning studied a general linear quadratic mean field type control problem by giving solution both in terms of a forward/backward system of stochastic differential equations and by a pair of Riccati equations. [25] further gave certain conditions so that the mean field type control problem is equivalent to a class of mean field game problems. [25] then gave a model of exhaustible resource production, and connected it to an equivalent linear quadratic mean field type control problem. By solving the associated pair of Reccati equations, equilibrium production and market price are obtained in explicit form.

Chan and Sircar [16, 17] showed that the mean-field equilibrium for production of exhaustible resources is the same for both Bertrand and Cournot type competition. Cournot competitions between exhaustible and renewable resources were studied in [17]. They first considered competition of producers, each of whom can produce costly renewable resources once exhaustible reserves run out. Then they studied competition of a large group of exhaustible producers with a single renewable producer, similar to the major-minor model of Huang [33]. It was found that when renewable production cost is high, the exhaustible producers may strategically increase production rate and hence lower the price. They also studied the impact of exploration and discovery in Cournot models of exhaustible resources and found that higher reserves lower exploration rates and increase production.

We apply the MFG approach to model energy markets with a continuum of producers of exhaustible resources. Through the mean field formulation, we aim to study equilibrium production quantities and reserves distribution in energy markets with a large population of competing producers. Particularly, we want to understand how exploration activities affect the long-term market organization, and how the exploration uncertainty permeates the solution. Three main quantities that we wish to investigate are: (i) the price effect of exploration, and related time-evolution; (ii) aggregate production implied by the model; (iii) aggregate exploration efforts. The related analysis yields quantitative insights into the macro behavior of major commodity markets, especially those for fossil fuels (crude oil, natural gas) where exhaustibility and associated E&P (Exploration and Production) activities are key to strategic behavior of the firms.

In comparison to previous studies and MFG models, our setup has several key differences. First, the price interaction coming from the aggregate production rate yields a non-standard mean-field coupling between the agent *controls* (rather than the more common state-space interaction), necessitating special treatment in the HJB equation. Second, the jump process modeling discrete reserves discoveries leads to a integral term in the HJB equation and a non-local term in the transport equation. Third, the hard constraint of exhaustibility (i.e. zero reserves) generates a non-standard, implicit boundary condition at $x = 0$ which again requires a tailored solution.

Our work fits into two different strands of game-theoretic models of energy production. On the one hand, we extend the works [16, 29] who considered exhaustible resources but without exploration; thus reserves were non-increasing. Specifically, the reserves processes in [29, 16] are modeled by Itô diffusion processes, and the transport equation of reserves distribution is determined by a standard Kolmogorov forward equation. In contra-distinction, in this chapter we consider exploration activities that stochastically lead to additional reserves. Thus, the reserves process is a controlled jump process generating a non-local, first-order transport equation. On the other hand, we extend the duopoly model [40]. In the latter model of an exhaustible producer and a renewable producer, each one has significant power of influence on price; in the MFG model herein, each producer has negligible power on market price that is rather driven by the *aggregate* production.

The closest work to ours is Chan and Sircar [17] who also considered an MFG setting with exploration. We give a more comprehensive investigation of exploration effects on game equilibrium in terms of total production, reserves distribution, and producers' behavior in limiting situations of exploration and discovery.

The above features generate several major challenges. For example, due to the jump process, the transport equation of the reserves distribution in our research is a *partial integro-differential equation* which leads to some more analytical and numerical difficulty. A major part of the paper is devoted to constructing an iterative numerical scheme to solve the MFG equations. Our numerical scheme decouples the HJB and transport equations via a Picard-like iteration that alternately updates the optimal production and exploration controls, and the reserves distribution function (which in turn determines the market price). Furthermore, since the HJB equation depends on time and space, we use *method of lines* to solve the HJB equation, that is, we discretize the HJB equation in the space dimension but not in time and solve the system of ordinary differential equations using fourth-order Runge Kutta method.

1.3 Model

1.3.1 Reserves Process

We consider a game model with N exhaustible producers (players). Each producer uses exhaustible resources, such as oil, to produce energy. Let X_t^i represents the reserves level of player i, $i = 1, ..., N$. Each X_t^i belongs to the state space of nonnegative real numbers $\mathbb{R}_+ \bigcup \{0\}$. Reserves level X_t^i decreases at a controlled production rate $q_t^i \geq 0$, and increases by jumps due to discovery of new resources. We use a controlled point process to model occurrences of new discoveries. Specifically, occurrences of new discoveries for player i are modeled through a point process N_t^i with controlled intensity $\lambda(t)a_t^i$, where a_t^i is the exploration efforts controlled by the player and $\lambda(t)$ is rate of discoveries per unit exploration effort. The parameter $\lambda(t)$ reflects the current exploration techniques and overall resources underground, hence it is taken as externally given and uniform for all producers. Since the total resources underground is decreasing due to exploration and production, it is reasonable to assume that $\lambda(t)$ is decreasing in time and

$$\lim_{t\to\infty} \lambda(t) = 0.$$

Let τ_n^i be the n-th arrival time of the point process N_t^i, then the inter-arrival time between the n-th and $(n+1)$-th arrivals satisfies the following probability distribution

$$\mathbb{P}\left(\tau_{n+1}^i > \tau_n^i + t\right) = \exp\left(-\int_0^t \lambda(s) a_{\tau_n^i+s}^i ds\right). \tag{1.1}$$

The positive quantity δ is the unit amount of a discovery, which is assumed to be constant as in [40, 41]. The unit amount δ of each discovery can be random in general, which we will address in the context of specific optimization problems. These upward jumps of fixed size δ mimic discrete discoveries of new oil fields (or perhaps new oil recovery technologies) that take place abruptly. Such "Poissonian" dynamics date back to work of Arrow and Chang [2] and are arguably more fidel to realistic reserves evolution. According to the setups above, the reserves dynamics of the players are given by the following stochastic differential equations

$$\begin{aligned} dX_t^i &= -q_t^i \mathbb{1}_{\{X_t^i>0\}} dt + \delta dN_t^i, \quad t > 0, \\ X_0^i &= x_0^i \geq 0, \quad i = 1, 2, ..., N. \end{aligned} \tag{1.2}$$

where each player i is assumed to have initial reserves level x_0^i, $i = 1, 2, ..., N$. The indicator function $\mathbb{1}_{\{X_t^i>0\}}$ implies that production shuts down whenever reserves run out, i.e., $X_t^i = 0$. The N players' reserves can be represented by a random vector defined as

$$\boldsymbol{X}_t^N := \left(X_t^1, \cdots, X_t^N\right).$$

The controlled dynamics (1.2) of the reserves can be described as *piecewise deterministic process* (PDP), because between discoveries no new information is coming in and reserves decrease continuously according to the production schedule. At discovery moments, reserves increase via an instantaneous jump of size δ.

1.3.2 Cost Functions

We assume that all producers have the same cost functions of production and exploration, denoted by $C_q(\cdot)$ and $C_a(\cdot)$, respectively,

$$C_q(q) = \kappa_1 q + \beta_1 \frac{q^2}{2}, \quad C_a(a) = \kappa_2 a + \beta_2 \frac{a^2}{2}. \tag{1.3}$$

The coefficients $\beta_{1,2}$ of the quadratic terms are assumed to be positive, which make the cost functions strictly convex for q and a large enough. Convexity of the cost functions guarantees that the optimal production and exploration effort levels are finite. The coefficients $\kappa_{1,2}$ of the linear terms represent constant marginal cost of production and exploration arisen from the use of facilities and labor. The coefficients $\beta_{1,2}$ of the quadratic terms represent increasing marginal cost proportional to quantities of production and exploration efforts. Production and exploration of exhaustible resources lead to negative externalities (such as rising labor costs or nonlinear taxation). We note that when $\kappa_2 = 0$ then exploration is ongoing, otherwise $a^* = 0$ could be optimal.

1.3.3 Price Determination

The market price p is determined from the global supply-demand equilibrium. The total demand is determined through a demand function $p \to D(p)$ which gives total demand levels at each price level p. The producers sell total quantities Q into the market and receive price p determined through a price function $p(Q)$. In the global supply-demand equilibrium, the price function should be equal to the inverse demand function $D^{-1}(Q)$, i.e. $p(Q) = D^{-1}(Q)$. We assume a linear price (or inverse demand) function

$$p = p(Q) = \bar{D} - Q, \tag{1.4}$$

where $Q := \sum_{i=1}^{N} q^i$ is the total production of all N producers, and $\bar{D}$ is the maximum (finite) *choke* price under zero supply. Producers interact with each other through this price mechanism that is driven by the aggregate supply (and affects player's profits) leading to non-cooperative game.

1.3.4 Game Value Functions and Strategies

In a continuous-time Cournot game model, each player chooses rate of production q^i in order to maximize profit which is equal to the revenue $p \cdot q^i$, minus the production and exploration costs, integrated and discounted at a rate $r > 0$. We work on a finite time horizon $[0, T]$, where T is exogenously specified. The role of the horizon will be revisited in the sequel. The price each player receives is determined through the inverse demand function (1.4).

We define the strategy profile $\boldsymbol{s}$ of all the N players by

$$\boldsymbol{s} := \left(s^1, s^2, ..., s^N\right),$$

with each player i's strategy vector denoted by $s^i := (q^i, a^i)$. Starting with initial reserves state $\mathbf{X_t} = \mathbf{x}$, each player's objective functional $\mathcal{J}^i, i = 1, \ldots, N$ on $[t, T]$

is defined as the total discounted profit

$$\mathcal{J}^i\left(\boldsymbol{s};t,\mathbf{x}\right) := \mathbb{E}\left\{\int_t^T\left[D^{-1}\left(\sum_{j=1}^N q_s^j\right)q_s^i - C_q(q_s^i) - C_a(a_s^i)\right]e^{-r(s-t)}\,ds\;\middle|\;\mathbf{X_t}=\mathbf{x}\right\}, \tag{1.5}$$

where the expectation is over the random point processes N^j that drive X^j and hence q^j's. Associated with the objective functional, we define player's i best response value function conditional on other players' strategies $\{s^{j,*}\}_{j\neq i}$ as the supremum

$$v^i(t,\mathbf{x}) := \sup_{s^i\in\mathcal{A}} \mathcal{J}^i\left(s^{1,*},\dots,s^{i-1,*},s^i,s^{i+1,*},\dots,s^{N,*};t,\mathbf{x}\right), \quad i=1,\dots,N. \tag{1.6}$$

We focus on the admissible set $\mathcal{A}$ of strategies whereby $s_t^i=(q_t^i,a_t^i)$ are Markovian feedback controls $q_t^i=q^i(t,\mathbf{X}_t)$, $a_t^i=a^i(t,\mathbf{X}_t)$ such that $\mathcal{J}^i(\mathbf{s};t,\mathbf{x})<\infty$, $\forall\mathbf{x}\in\mathbb{R}_+^N$, for all $i=1,\dots,N$. The exhaustibility constraint also imposes that $q^i(t,0)\equiv 0$ is the only admissible control.

From (1.5) and (1.6), we see that each player's choice of strategy depends on the strategies of all the others. We aim to study the Nash equilibrium of this game. To be more precise, we look for Markov feedback Nash equilibrium, because the strategies set from which the players choose is of Markov feedback form.

Definition 1.1 (Nash equilibrium of N-player game)**.** *The Nash equilibrium of N-player game is a strategy profile $\boldsymbol{s}^*=\left(s^{1,*},\dots,s^{N,*}\right)$ of the N players with each*

$s^{i,*} = (q^{i,*}, a^{i,*})$ *such that*

$$\mathcal{J}^i\left(\boldsymbol{s}^*; t, \mathbf{x}\right) \geq \mathcal{J}^i\left((\boldsymbol{s}^{*,-i}, s^i); t, \mathbf{x}\right), \quad \forall i \in \{1, 2, \ldots, N\}, \tag{1.7}$$

where $\boldsymbol{s}^{*,-i}$ *is the strategy profile* $\boldsymbol{s}^*$ *with the i-th entry* $s^{i,*}$ *replaced by arbitrary* $s^i = (q^i, a^i) \in \mathcal{A}$.

In words, a Nash equilibrium is the set of strategies of the N players such that no one can better off by unilaterally changing his own strategy. The feedback structure of the controls $q_t^i = q^i(t, \mathbf{X_t})$, $a_t^i = a^i(t, \mathbf{X_t})$ together with (1.4) imply that player i's dependence on $\mathbf{X_t}$ can be summarized by his individual reserves X_t^i and the aggregate distribution of all players' reserves. The latter is characterized through the upper-cumulative distribution function defined by

$$\eta^N(t, x) := \frac{1}{N} \sum_{j=1}^{N} \mathbb{1}_{\left\{X_t^j \geq x\right\}}. \tag{1.8}$$

Thus the Markovian feedback controls (q^i, a^i) can be equivalently represented as

$$q_t^i = q^i\left(t, X_t^i; \eta^N(t, \cdot)\right), \quad a_t^i = a^i\left(t, X_t^i; \eta^N(t, \cdot)\right), \qquad i = 1, \ldots, N. \tag{1.9}$$

Theoretically the Nash equilibrium of the N-player game can be found by Hamilton-Jacobi-Isaacs (HJB-I) approach. HJB-I approach is to use dynamic programming principle to derive the partial differential equation of each player's game value function, with other players' strategies as entries. It is extremely hard to find a Nash equilibrium by using the (HJB-I) approach either analytically or numerically, even for small N, e.g. $N = 2$. In Chapter 2 we study the Cournot game between an exhaustible producer and a renewable producer in high and low demand regimes, which involves a system of equations with dimensionality equal to two (number of players) by two (number of demand regimes). The renewable producer has infinite resources, but has positive fixed production costs. The production strategy of the renewable producer depends on the production of the exhaustible producer. Only the exhaustible producer has controls on the reserves level as the unique state variable. Hence the equations of the exhaustible producer are autonomous from the equations of the renewable producer, and it suffices to deal with the equations of the exhaustible producer to obtain the Nash equilibrium of the game. Moreover, We consider the stationary case in which exploration can go forever, thus we just need to deal with a system of ordinary differential equations.

In Chapters 3 and 4, we employ mean filed game approach to find approximate Nash equilibrium of Cournot game involving a continuum of players ($N \to \infty$), which reduces the system of N coupled equations in HJB-I framework to two doubly coupled equations: one is the HJB equation of a representative player; and the other is the equation of the evolution of distribution of all the players.

Chapter 2

Dynamic Cournot Game Under Stochastic Demand

2.1 Model Overview

In this chapter we study dynamic Cournot games between two players: producer 1 that extracts a non-renewable resource (oil) and has to worry about reserves; and producer 2 who extracts a renewable resource (green energy) and therefore has infinite reserves. We also consider two stochastic state variables: demand D_t and current reserves X_t of the exhaustible player 1. Moreover, agents have a total of three controls, namely production rates for producers 1 and 2, as well as exploration effort for producer 1.

The two producers compete through a Cournot framework, in which producers choose quantities of energy to produce and receive profit based on a single market price determined through aggregate supply. Costs of the exhaustible player are driven solely by the costly (convex) exploration efforts; her production costs are taken to be zero. On the contrary, the green producer has a positive marginal cost of production but inexhaustible resources so his additional shadow marginal costs are zero. These production costs of player 2 are a proxy for the amount of competition. They also reflect the present reality of non-renewable energy production as being the cheaper incumbent against the new renewable entrants. The Cournot framework is used because at the macro level, energy is perfectly substitutable and so the two producer's products are in direct competition.

Our aim is to study this duopoly of the exhaustible resources producer with a green producer in terms of dynamic Nash eqilibria. The model is cast in continuous-time so as to allow use of the well-understood Hamilton-Jacobi-Bellman-Isaacs (HJB-I) methodology that reduces computational analysis to study of coupled systems of differential equations. In order to simplify the mathematics as much as possible while maintaining dynamic effects, we keep the dynamics of (D_t) and (X_t) stylized, To this end, reserves (X_t) follow piecewise deterministic trajectories, smoothly decreasing due to production and experiencing constant-size jumps upon new reserves discovery.

The demand level (D_t) is modeled as $D_t = D(M_t)$ where M_t is a finite-state Markov chain; this is meant to evoke the popular regime-switching models that are frequently used in financial mathematics to model the business cycle fluctuations. In the context of single-agent optimization, a related model of resource extraction (with an exogenous discovery process) within a random environment was studied in [22].

The main setting just takes D_t to have two possible levels L, H. The demand level modulates the common price obtained by the producers for a fixed supply level. In a toy setting this occurs linearly. Both state variables are modeled by stationary processes, leading to an infinite-horizon discounted game. With the Markovian dynamics, this allows to reduce equilibrium behavior to Markov feedback (closed-loop) strategies. A drawback is that agents are infinitely long-lived (i.e. never leave or enter the game) and no off-equilibrium behavior is modeled.

The combination of the above choices keeps the overall state-space as simple as possible and in particular makes the HJB-I equations first-order only, removing many of the analytical difficulties arising in second-order equations (for example [31] found that these equations can sometimes be hyperbolic rather than parabolic, causing unstable analytic and numeric behavior). Moreover, since only a single agent has reserves, there is just one continuous state-variable, effectively allowing us to deal only with ordinary differential equations, rather than partial differential equations. We however stress that exploration necessarily introduces additional subtleties; in our model it brings in a *non-local* term that requires careful treatment even at the implementation level.

2.2 Dynamic Cournot game under stochastic demand

Two players (named 1 and 2) produce perfectly substitutable goods at rates q^1, q^2. The price p is determined by the price (inverse demand) function (1.4) introduced in section 1.3.3 with total quantity of production $Q = q^1 + q^2$. In the present chapter we consider the situation of *stochastic demand*, whereby market demand exhibits exogenous fluctuations over time. All variables are thereafter continuously indexed by $t \in \mathbb{R}_+$. We model stochastic demand by making $\bar{D}$ non-constant, modulated by an exogenous factor (M_t), namely

$$p_t \equiv p(q^1, q^2, M_t) = M_t - q^1 - q^2. \tag{2.1}$$

We assume that (M_t) is a finite-state stationary Markov chain with state space E and generator $\Lambda \equiv (\lambda_{ij})$. Thus, a larger value of M_t means stronger demand and therefore a higher price p_t for the same level of supply. For illustrative purposes we shall focus on the case where (M_t) is a two-state Markov chain with state space $\{D_0, D_1\} \equiv \{L, H\}$, where $0 < L \leq H$. In that case we label the time-homogeneous switching rates between the two regimes of (M_t) as λ_{01} and λ_{10} respectively.

Player 1 extracts a non-renewable resource that may become exhausted. His reserves at time t are denoted by $X_t \geq 0$. Reserves decrease through production but can be replenished via exploration. Without any reserves, the player may not produce but may continue to search for replenishments. Denote by $a_t \geq 0$ the exploration effort at time t, and let (N_t) be a point process for counting discoveries of new resources. Then (N_t) has controlled intensity λa_t, and the arrival times τ_n's of (N_t) satisfy (1.1). The unit amount of new discovery is a fixed $\delta > 0$. Overall, the reserve process (X_t) of player 1 follows the dynamics (1.2) for a single producer $i = 1$, with q_t^1 being the production rate. Exploration is costly and generates costs at rate $C_a(a_t)$ per unit time, where the cost function takes the function form $C_a(a) = \kappa_2 a + \beta_2 \frac{a^2}{2}$ given by (1.3).

Player 2 always has infinite resources, but faces positive fixed production costs $c \geq 0$. It is possible for the controls to be zero in which case there is no production (reserves remain constant) or no exploration (i.e. discovery rate is zero).

Players aim to maximize their total discounted profit, which is equal to the instantaneous revenue $p_t \cdot q_t$, minus the production and exploration costs, integrated and discounted (using continuous discount rate $r > 0$) on the infinite time horizon.

To analyze the game equilibria we use the notion of Markov Nash equilibria. Thus, player strategies are assumed to be in closed-loop feedback form, $q_t^\ell = q^\ell(X_t, M_t)$, $\ell = 1, 2$ and $a_t = a(X_t, M_t)$. Given an equilibrium $(q^{\ell,*}(X_t, M_t), a^*(X_t, M_t))$ we denote the corresponding game functions of producer 1 by $v_L(x)$ and $v_H(x)$; and the game functions of producer 2 by $g_L(x)$ and $g_H(x)$. Here the subscript indicates the initial value $M_0 \in \{L, H\}$ of the Markov chain. These game values are the discounted cumulative expected profits starting with $X_0 = x, M_0 = i$,

$$v_i(x) = \mathbb{E}\left[\int_0^\infty e^{-rt}\left(q_t^{1,*}p(q_t^{1,*}, q_t^{2,*}, M_t) - C_a(a_t^*)\right)\,dt\middle|X_0 = x, M_0 = i\right];$$
$$g_i(x) = \mathbb{E}\left[\int_0^\infty e^{-rt}q_t^{2,*}\left(p(q_t^{1,*}, q_t^{2,*}, M_t) - c\right)\,dt\middle|X_0 = x, M_0 = i\right],$$

and must satisfy the Nash optimality conditions

$$v_i(x) = \sup_{q^1,a}\mathbb{E}\left[\int_0^\infty e^{-rt}\left(q_t^1 p(q_t^1, q_t^{2,*}, M_t) - C_a(a_t)\right)\mathbb{1}_{\{X_t>0\}}\,dt\middle|X_0 = x, M_0 = i\right]$$
$$g_i(x) = \sup_{q^2}\mathbb{E}\left[\int_0^\infty e^{-rt}q_t^2\left(p(q_t^{1,*}, q_t^2, M_t) - c\right)\,dt\middle|X_0 = x, M_0 = i\right], \quad i = L, H. \tag{2.2}$$

Thus, given the other player's equilibrium strategy, each player chooses optimal strategies for her own production (and exploration). To analyze (2.2), we employ the Hamilton-Jacobi-Bellman-Isaacs framework that aims to express game values through a system of coupled differential equations. Define $\Delta f(x) := f(x + \delta) - f(x)$. We also index a generic regime by $i = 1, 2$ and the *other* regime by j.

Assuming the functional forms of demand in (2.1) and exploration costs in (1.3), the HJB-I ordinary differential equations of v_L, v_H and g_L, g_H are

$$\sup_{q_i^1}\left[q_i^1(x)\left(D_i - q_i^1(x) - q_i^{2,*}(x)\right) - v_i'(x)q_i^1(x)\right] + \sup_{a_i}\left[a_i\lambda\Delta v_i(x) - C_a(a_i)\right]$$
$$+ \lambda_{ij}(v_j(x) - v_i(x)) - rv_i(x) = 0, \tag{2.3}$$

$$\sup_{q_i^2}\left[q_i^2(x)\left(D_i - q_i^{1,*}(x) - q_i^2(x) - c\right)\right] - g_i'(x)q_i^{1,*}(x) + a_i^*(x)\lambda\Delta g_i(x)$$
$$+ \lambda_{ij}(g_j(x) - g_i(x)) - rg_i(x) = 0. \tag{2.4}$$

Upon exhaustion of reserves $X_t = 0$, player 1 can no longer produce, yielding a temporary monopoly for player 2. However, player 1 remains in the game and may continue to explore for reserves (financing exploration by borrowing against future earnings). Fix $X_0 = 0$ and denote by $\tau \equiv \tau_1$ the time of the first discovery of new reserves (so that $X_t = 0$ on $[0, \tau)$ and $X_\tau = \delta$) and by σ the first transition time of the Markov chain (M_t). Then by conditioning on τ and σ we have

$$v_i(0) = \sup_{a_i \geq 0} \mathbb{E}\left\{ \mathbb{1}_{\{\sigma<\tau\}}\left[e^{-r\sigma}v_j(0) - \int_0^\sigma e^{-rt}C_a(a_i)\,dt\right]\right. \tag{2.5}$$
$$\left. + \mathbb{1}_{\{\tau\leq\sigma\}}\left[e^{-r\tau}v_i(\delta) - \int_0^\tau e^{-rt}C_a(a_i)\,dt\right] \;\Big|\; X_0 = x, M_0 = i\right\} \vee 0.$$

By stationarity of (M_t) it follows that the optimal exploration rate a_i is constant until $\tau\wedge\sigma$ and hence $\tau\wedge\sigma \sim Exp(\lambda a + \lambda_{ij})$ has an exponential distribution. Using the fact that $\mathbb{P}(\tau < \sigma) = \frac{\lambda a_i}{\lambda a_i + \lambda_{i\bar{i}}}$ then leads to

$$v_i(0) = \sup_{a_i\geq 0} \frac{v_j(0)\lambda_{ij} + v_i(\delta)\lambda a_i - C_a(a_i)}{r + \lambda_{ij} + \lambda a_i} \vee 0, \tag{2.6}$$

yielding an implicit condition linking $v_i(0), v_i(\delta)$ and $v_j(0)$.

Optimizing for the production rates q^ℓ which must be non-negative in (2.3)-(2.4) yields that the candidate equilibrium strategies are given by

$$\begin{cases} q_i^{1,*}(x) = \dfrac{1}{2}\max\left(D_i - q_i^{2,*}(x) - v_i'(x), 0\right), \\ q_i^{2,*}(x) = \dfrac{1}{2}\max\left(D_i - q_i^{1,*}(x) - c, 0\right). \end{cases} \tag{2.7}$$

For simpler notation, we write $z^+ \equiv \max(z, 0)$. Figure 2.1 illustrates how (2.7) is used to determine the equilibrium given a fixed value of say $v_H'(x)$. Assuming $C(a) = \kappa_2 a + \beta_2 \frac{a^2}{2}$, the candidate optimal exploration rate is similarly

$$a_i^*(x) = [(\lambda \Delta v_i(x) - \kappa)^+]. \tag{2.8}$$

Equation (2.8) holds also for $x = 0$ since the exhaustibility constraint does not apply to exploration.

We observe that (2.3) yields two coupled equations for $v_L(x)$ and $v_H(x)$ which are however autonomous from $g_i(x)$. This is due to the state variable (X_t) being completely controlled by player 1. The system (2.3) features only a first-order differential of $v_i(x)$ due to the continuous decrease in (X_t); it also has two non-local effects, the term $\Delta v_i(x)$ arising from jumps induced by exploration successes, and the term $v_j(x) - v_i(x)$ due to the regime-shifts in (M_t). Therefore, overall (2.3) is a system of first-order nonlinear (forward)-delay ODEs in x. Respectively, (2.4) leads to a first order linear delay-ODE for $g_L(x)$ and $g_H(x)$ in terms of $v_L(x)$

and $v_H(x)$. However, as already written in (2.7), all the equilibrium production and exploration strategies depend only on $v_i(x)$, and so we will not deal much with the g_i equations, focusing mostly on (2.3).

2.2.1 Game Stages

The maximizers in (2.3)-(2.4) intuitively determine the equilibrium strategies of the players. Several different equilibrium types are possible due to the constraints $q^1 \geq 0$, $q^2 \geq 0$ and $a \geq 0$ that can be binding. These situations can be seen through the piecewise nature of (2.3)-(2.4) that arises from the $\max(\cdot, 0)$ terms. They can also be imagined through Figure 2.1: if the two piecewise linear curves for $q_i^{1,*}, q_i^{2,*}$ do not cross in the interior then we have a boundary solution on one of the axes.

For each L, H, and c fixed, the strategies of the two players depend on the shadow reserves cost $v_i'(x), i = L, H$, which determines the type of equilibrium at $X_t = x$. Depending on the shadow costs, the alternative game types are:

Type I: *Interior* solution where both players are active: $q^1 > 0, q^2 > 0$. This case arises when $v_i'(x)$ satisfies $2c - D_i < v_i'(x) < \frac{c+D_i}{2}$.

Type M1: The exhaustible player 1 has a monopoly because $q^2 = 0$. This occurs when $v_i'(x) \leq 2c - D_i$.

Type M2: The green player 2 has a monopoly while $q^1 = 0$. This occurs when

$v_i'(x) \geq \frac{c+D_i}{2}$.

In each case we further can have either $a_i(x) > 0$, or $a_i(x) = 0$ (no exploration, i.e. "saturation" of reserves) depending on the positivity of the term $\Delta v_i(x) - \kappa$ in (2.8).

In the interior game Type I, we have

$$q_i^1 = \frac{1}{3}(D_i + c - 2v_i'(x)) \qquad q_H^2 = \frac{1}{3}(D_i + v_i'(x) - 2c), \quad x > 0, \tag{2.9}$$

and the ODE of v_i reduces to

$$v_i'(x) = \frac{D_i + c}{2} - \frac{3}{2}\left[(\lambda_{10} + r)v_i(x) - \lambda_{ij}v_j(x) - \frac{1}{\gamma}((\lambda\Delta v_i(x) - \kappa)^+)^\gamma\right]^{\frac{1}{2}}. \tag{2.10}$$

In Type M1 equilibrium, exhaustible producer 1 monopolizes the market,

$$q_i^1 = \frac{1}{2}\left(D_i - v_i'(x)\right), \quad q_i^2 = 0,$$

and the ODE of v_i is given by the monopoly equation

$$\frac{1}{2}(D_i - v_i'(x))^2 + \frac{1}{\gamma}((\lambda\Delta v_i(x) - \kappa)^+)^\gamma + \lambda_{ij}(v_j(x) - v_i(x)) - rv_i(x) = 0. \tag{2.11}$$

In Type M2 equilibrium, the green producer 2 monopolizes the market, $q_i^1 = 0$, giving $q_i^2 = \frac{1}{2}\left(D_i - c\right)$, and $v_i(x)$ is determined through the nonlinear equation

$$\frac{1}{\gamma}((\lambda\Delta v_i(x) - \kappa)^+)^\gamma + \lambda_{ij}v_j(x) - (r + \lambda_{ij})v_i(x) = 0. \tag{2.12}$$

Intuitively, $v_i(x)$ is concave, so that $x \mapsto v_i'(x)$ is decreasing. Thus, Type M2 equilibrium arises for small x (when the shadow cost of exhaustibility is very large, driving player 1 to sit out); Type I equilibrium arises for moderate x and Type M1 equilibrium arises for large x where the shadow cost is negligible. We note that in high demand regime, exhaustible resources production is always positive for any $x > 0$ since player 1 cannot expect higher profit by holding on to reserves. Therefore Type M2 equilibrium may only arise for small x and $M_t = L$.

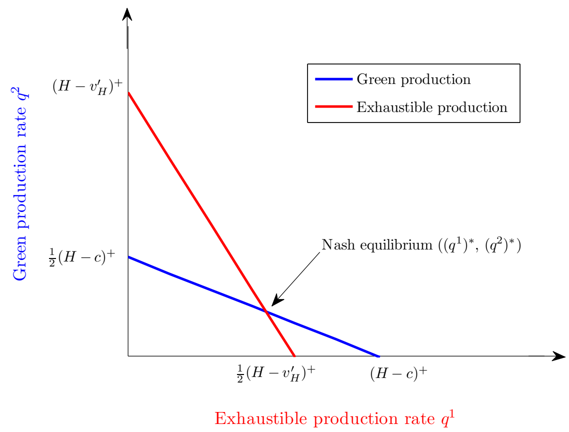

Figure 2.1: Nash equilibrium of the Cournot duopoly in high-demand regime. The two piecewise linear curves show optimal production rates of player 1 and player 2 given $v_H'(x)$ and the production rate of the other player (e.g. $q^{1,*}(x; q^2(x), v_H'(x))$) as defined in (2.7). Equilibrium is achieved when the curves cross.

Combining (2.10)-(2.12), the HJB ODEs of $v_i(x)$ can be written as the single piecewise-defined equation

$$\left[\frac{2}{3}\left(\frac{D_i+c}{2}-v_i'(x)\right)^+ - \frac{1}{6}\left(2c-D_i-v_i'(x)\right)^+\right]^2 + \frac{1}{\gamma}\left[(\lambda\Delta v_i(x)-\kappa)^+\right]^\gamma$$
$$+\lambda_{ij}v_j(x)-(\lambda_{ij}+r)v_i(x)=0, \quad i,j=L,H. \quad (2.13)$$

Remark 2.1. *Within a single-agent optimization setting, [22] showed the well-posedness of the system* (2.10) *and associated implicit boundary condition* (2.22)*. In particular, using the functional equations methods of [46], Deshmukh and Pliska [22] prove that* (2.10) *has a unique bounded differentiable solution and the optimal controls are indeed* (2.9)*. They moreover show that v_i is strictly concave, i.e. marginal cost of reserves is decreasing in x. However, these analytic tools become unavailable in the presence of free boundaries that arise due to game effects such as blockading or production shut-down (i.e. other equilibria types beyond Type I). In particular, the interaction of the non-local term $\Delta v_i(x)$ with the piecewise-defined functional form of $v_i'(x)$ in* (2.13) *(and the additional coupling among $v_i(x)$ and $v_j(x)$) poses a major challenge. As a result, we are not able to analytically establish smoothness of* (2.13) *in the duopoly model.*

2.2.2 Numerical Scheme

Due to the challenge of implicit boundary condition and the presence of a "forward" delay term on the semi-infinite domain $\mathbb{R}_+$, numerically solving the system (2.3) (or (2.13)) is nontrivial. We note that the equations have a non-local term and several free boundaries indicated by the critical values of x that trigger the $(\cdot)^+$ terms to be zero. In particular, there is x_i^{start} which separates Type M2 and Type I equilibria; x_i^{block} that separates Type I and Type M1 equilibria; and x_i^{sat} that separates regions where $a_i(x) > 0$ and $a_i(x) = 0$. The meaning of these quantities is the production start level (below x_i^{start} production is shut-down); the blockading level (above x_i^{block} player 2 is blockaded) and the reserves saturation level (above x_i^{sat} no exploration takes place).

To solve the HJB-I ODE system we use the iterative scheme in [40] that originates in the ideas of Davis [20]. Let $v_L^{(0)}(x)$ and $v_H^{(0)}(x)$ be the game values corresponding to the case where resources are completely non-replenishable, so that no new resource discoveries are possible. Similarly to (2.13) (and removing the exploration-related term), we have that

$$\left[\frac{2}{3}\left(\frac{D_i + c}{2} - (v_i^{(0)})'(x)\right)^+ - \frac{1}{6}\left(2c - D_i - (v_i^{(0)})'(x)\right)^+\right]^2 + \lambda_{ij} v_j^{(0)}(x) - (\lambda_{ij} + r) v_i^{(0)}(x) = 0, \quad i, j = L, H, \quad (2.14)$$

with boundary conditions $v_L^{(0)}(0) = v_H^{(0)}(0) = 0$, since starting with no reserves ($x = 0$) and no possibility of discoveries, player 1 will never have any revenue. We then define inductively for $n \geq 1$ the functions $v_i^{(n)}$ via

$$\left[\frac{2}{3}\left(\frac{D_i + c}{2} - (v_i^{(n)})'(x)\right)^+ - \frac{1}{6}\left(2c - D_i - (v_i^{(n)})'(x)\right)^+\right]^2$$
$$+ \frac{1}{\gamma}\left[\left(\lambda(v_i^{(n-1)}(x+\delta) - v_i^{(n)}(x)) - \kappa\right)^+\right]^\gamma + \lambda_{ij} v_j^{(n)}(x) - (\lambda_{ij} + r) v_i^{(n)}(x) = 0, \tag{2.15}$$

with boundary conditions

$$v_i^{(n)}(0) = \sup_{a_i \geq 0} \frac{v_j^{(n)}(0)\lambda_{ij} + v_j^{(n-1)}(\delta)\lambda a_i - C_a(a_i)}{r + \lambda_{ij} + \lambda a_i}, \quad i, j = L, H. \tag{2.16}$$

Note that (2.15)-(2.16) partially uncouple the original equations by making the non-local term a source term instead. Therefore, (2.15)-(2.16) are now a standard system of nonlinear first order ODE with an implicit boundary condition at $x = 0$. We start the computation with no discovery case $n = 0$ by solving the system (2.14). For $n \geq 1$, we solve the system (2.15) by using the data from the $(n-1)$-st iteration for the forward delay term in the n-th iteration. In numerical computation, at each $x > 0$ we start with assuming $v_i'(x)$ satisfy any one of the three equilibrium types and then solve for $v_i(x)$ and $v_i'(x)$. If the computed $v_i'(x)$ is consistent with the assumed range of values, the assumption and the results are correct, otherwise we switch to another assumption to compute $v_i(x)$ and $v_i'(x)$.

Remark 2.2. *The equation* (2.15) *is an implicit quadratic in* v'. *To use a basic time-stepping ODE solver then requires inverting this to directly express* $v'(x)$ *in terms of* $(x, v(x))$. *Due to the special form above, one obtains* $v'(x) = \sqrt{F(x, v(x))}$ *where only the positive square root is relevant since* $v(x)$ *must be increasing. Standard tools such as fourth-order Runge-Kutta method can be applied to solve the ODEs. We use Matlab ODE solver function* `ode45` *to solve the ODEs in the research.*

Informally, $v^{(n)}$ represents the game value assuming a horizon of τ_n, where τ_n is the time of n-th resource discovery. It follows that the absolute error $|v_i^{(n)}(x) - v_i(x)|$ is bounded by $C\mathbb{E}[e^{-r\tau_n}]$. In turn, using the fact that exploration rates are uniformly bounded (in particular $a_i^*(x) \leq a_H^*(0) \forall x$) we have by a simple coupling argument that $\tau_n > e_n$ (in the sense of first-order stochastic dominance) where $e_n \sim Gamma(n, \lambda a_H^*(0))$, so that $\mathbb{E}[e^{-r\tau_n}] = O(C^n)$ for some constant $C < 1$. It follows that $v^n \to v$ exponentially fast. Practically, we have observed convergence after 10-20 iterations in all examples presented.

Remark 2.3. *The size* δ *of each new discovery can be random in general. Namely, we may model discovery amounts via a stochastic sequence* δ_n, $n = 1, 2, ...$, *where each* δ_n *is identically distributed with some distribution* $F_\delta(\cdot)$ *and independent of everything else in the model (in particular of other* δ_n*'s and of past controls). Introducing* F_δ *entails replacing* $v_i(x + \delta)$ *in the HJB-I equations and boundary* *conditions with $I_\delta[v_i](x) := \int_0^\infty v_i(x+u)F_\delta(du)$. Similarly, one would apply this integral operator I_δ to $v^{(n-1)}$ in which is straightforward to handle numerically.*

2.2.3 Illustration

Figure 2.2 illustrates the obtained solution of (2.3)-(2.4). We show the game values, production rates and exploration rates for two different values of player 2 costs c so as to illustrate the basic impact of competition. As expected, $v_i(x)$ are concave increasing; $q_i^1(x)$ are also concave increasing; $q_i^2(x)$ are convex decreasing, and $a_i(x)$ is convex decreasing (note that in general the control mappings will not be differentiable in x across the free boundaries so the above characterization is heuristic). Note that the impact of c is ambiguous. While lower c raises competition, it may also spur higher exploration efforts since the marginal cost of reserves could rise. Hence $c \mapsto a_i(x;c)$ may be non-monotone. In contrast, competition unambiguously lowers production rates $q_i^1(x)$ of the exhaustible player and her game value $v_i(x)$.

In the far-field limit $x \to \infty$, dependence on reserves vanishes and all quantities have a limit that can be directly computed via a one-stage static game. In particular, we have

$$\lim_{x\to\infty} v_L(x) = \frac{1}{r}\frac{(\lambda_{10}+r)\left(\frac{L+c}{2}-\frac{1}{6}(2c-L)^+\right)^2+\lambda_{01}\left(\frac{H+c}{2}-\frac{1}{6}(2c-H)^+\right)^2}{r+\lambda_{01}+\lambda_{10}};$$

$$\lim_{x\to\infty} v_H(x) = \frac{1}{r}\frac{\lambda_{10}\left(\frac{L+c}{2}-\frac{1}{6}(2c-L)^+\right)^2+(\lambda_{01}+r)\left(\frac{H+c}{2}-\frac{1}{6}(2c-H)^+\right)^2}{r+\lambda_{01}+\lambda_{10}}.$$

Similarly, the limiting values of the controls admit explicit solutions, including $\lim_{x\to\infty} a_i(x) = 0$.

2.3 Effects of Stochastic Demand on Production and Exploration

Stochastic demand is one of the key features of our model which is determined by two characteristics: the demand levels D_i and stationary distribution (π_L, π_H) of the demand regimes that is driven by the switching rates λ_{ij}. We analyze the effects of changes in these parameters on the equilibrium strategies of production and exploration.

The regime-switching stochastic demand is meant to mimic the macroeconomic business cycle. When the macroeconomy is running low, the demand for energy is low; when the macroeconomy is running high, the demand for energy is high and therefore the price function moves up to the high regime. In general, higher demand (or better opportunities for profit), lead to higher value for the

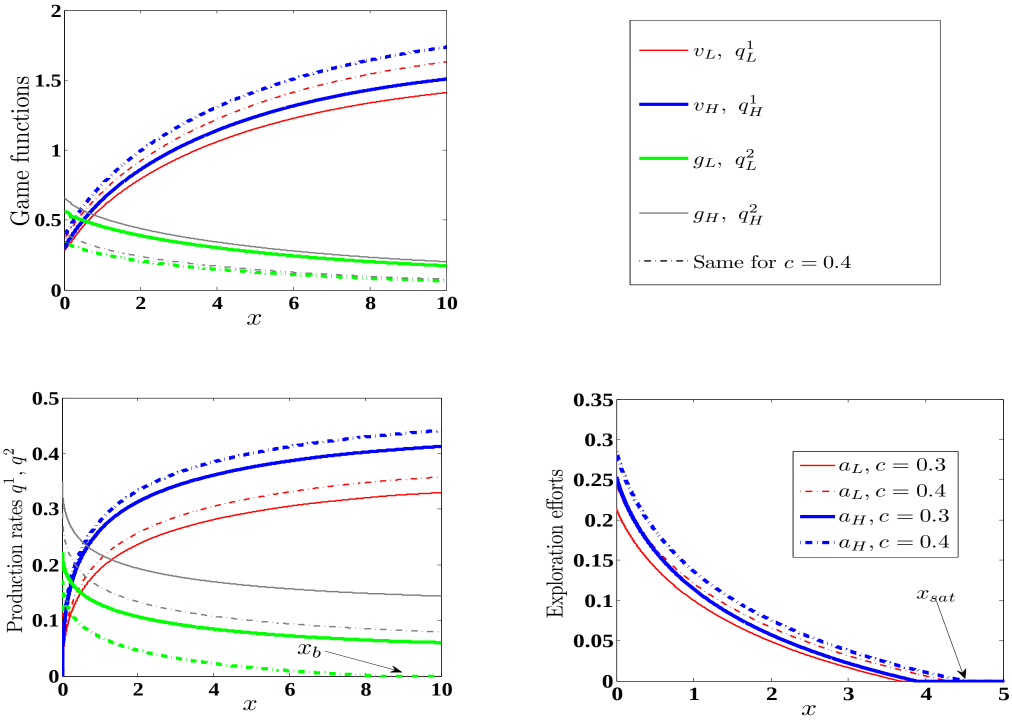

Figure 2.2: Duopoly axis game solutions for two different levels of production cost c: solid curves are for $c = 0.3$; dashed for $c = 0.4$. Tope left panel: Game functions $v_i(x), g_i(x)$. Bottom left panel: Production rates $q_i^\ell(x)$, $\ell = 1, 2$. Bottom right panel: Exploration efforts $a_i(x)$ of exhaustible producer 1. We take linear inverse demand with $L = 0.75, H = 1$, and switching rates $\lambda_{01} = \frac{1}{3}, \lambda_{10} = \frac{1}{5}$. Exploration costs are $C_a(a) = 0.1a + a^2/2$.

producer and induce both higher production and higher exploration. This is because the marginal value of reserves rises, stimulating both short-term production and longer-term extraction. This basic feature is present in all single-parameter comparative statics we tried, such as unilaterally changing the rates $\lambda_{01}, \lambda_{10}$ or increasing/decreasing the levels of market demand L, H. Similarly, due to the intuitive ordering of the two regimes, we expect that changing the regime to high while keeping all other parameters constant should also increase expected profit, increase production and increase exploration. Indeed, by a coupling argument it is easy to show that $v_H(x) > v_L(x)$. It is also interesting to note the difference between $v'_H(x)$ and $v'_L(x)$. When $x = 0$, the game is Type M2 ($q^1_L(x) = q^1_H(x) = 0$) which satisfies that $v'_L(x) \geq \frac{L+c}{2}$, $v'_H(x) = \frac{H+c}{2}$, and thus $v'_H(x) - v'_L(x) \leq \frac{H-L}{2}$. When $x > 0$ is large enough, the game becomes Type M1 such that $v'_L(x) = v'_H(x) = 0$ which implies $v'_H(x) - v'_L(x) = 0$. Since the marginal values $v'_L(x)$, $v'_H(x)$ of the two regimes are decreasing in x, we can expect that the difference $v'_H(x) - v'_L(x)$ is also decreasing in x. The conjecture is summarized as follows.

Conjecture 1. *Suppose that the state space of (M_t) is $E = \{L, H\}$. Then for all x,*

$$0 \leq v'_H(x) - v'_L(x) \leq \frac{H-L}{2}. \tag{2.17}$$

Corollary 1. *Suppose Conjecture 1 holds. Then for all $x \geq 0$, $0 \leq q_H^1(x) - q_L^1(x) \leq \frac{H-L}{2}$ and $a_H(x) > a_L(x)$.*

Proof. Substituting the conjectured relationship of v_L' and v_H' into the equations for $q_i^1(x)$ gives the stated inequalities (note that the upper bound applies in Type M1 and M2 equilibria only; a tighter bound is possible under Type I equilibrium). Similarly, we have

$$\Delta v_L(x) = v_L(x+\delta) - v_L(x) = \int_0^\delta v_L'(x+u)du$$
$$\leq \int_0^\delta v_H'(x+u)du = v_H(x+\delta) - v_H(x) = \Delta v_H(x),$$

and therefore $a_L^*(x) = [(\lambda \Delta v_L(x) - \kappa)^+]^{\gamma-1} \leq [(\lambda \Delta v_L(x) - \kappa)^+]^{\gamma-1} = a_H^*(x)$. □

Numerically, we have observed Conjecture 1 holding in all parameter settings we tried; but in general it is known that comparison of stochastic regimes is extremely difficult (see an extended discussion on this issue in [22]). Finally, we note that Conjecture 1 only covers two regimes and the situation where (M_t) modulates $p(\vec{q}_t)$ only. See Section 2.4.1 for counterexamples in more general situations.

2.3.1 Limiting Cases

The stationary distribution of the Markov chain (M_t) driving demand regimes is

$$\pi_L = \frac{\lambda_{10}}{\lambda_{01} + \lambda_{10}}, \quad \pi_H = \frac{\lambda_{01}}{\lambda_{01} + \lambda_{10}}. \tag{2.18}$$

The game value functions v_L, v_H of producer 1 under stochastic demand are bounded above by the value function under the high regime demand $D_t \equiv H$, which we denote by $\tilde{u}_H$. They are also bounded below by the value function under the deterministic low-regime demand $D_t \equiv L$, which we denote by $\tilde{u}_L$. As a result, v_L, v_H are "averages" of $\tilde{u}_L$ and $\tilde{u}_H$. The following two Lemmas clarify this interpretation by considering the extreme cases.

Lemma 2.1. *We have* $\lim_{\lambda_{01}\to+\infty} v_L(x) = \lim_{\lambda_{01}\to+\infty} v_H(x) = \tilde{u}_H$*, the game value of the model with fixed demand* $D_t \equiv H$*.*

Proof. See the Appendix. □

Lemma 2.2. *Let* $\lambda_{01} = \frac{c_{01}}{\epsilon}, \lambda_{10} = \frac{c_{10}}{\epsilon}$*. As* $\epsilon \to 0$*, the game value functions* $v_L(x)$ *and* $v_H(x)$ *both converge to* $\bar{v}(x)$*, which is determined by the following delay ODE*

$$\begin{aligned}\pi_L &\left[\frac{2}{3}\left(\frac{L+c}{2} - \bar{v}'(x)\right)^+ - \frac{1}{6}\left(2c - L - \bar{v}'(x)\right)^+\right]^2 \\ &+ \pi_H \left[\frac{2}{3}\left(\frac{H+c}{2} - \bar{v}'(x)\right)^+ - \frac{1}{6}\left(2c - H - \bar{v}'(x)\right)^+\right]^2 \\ &+ \frac{1}{\gamma}[(\lambda\Delta\bar{v}(x) - \kappa)^+]^\gamma - r\bar{v}(x) = 0, \quad x > 0\end{aligned} \tag{2.19}$$

with boundary condition

$$\bar{v}(0) = \frac{\lambda\bar{a}(0)\bar{v}(\delta) - C_a(\bar{a}(0))}{r + \lambda\bar{a}(0)}, \quad \bar{a}(0) = [(\lambda\Delta\bar{v}(0) - \kappa)^+]^{\gamma-1}, \tag{2.20}$$

where π_i *are given in* (2.18).

Proof. See the Appendix. □

The equation (2.19) can have as many as 5 free boundaries due to the multiple piecewise defined terms $(\cdot)^+$. While we have the basic ordering $(L+c)/2 < (H+c)/2$ and $2c - H < 2c - L$, the relationship between say $(H+c)/2$ and $2c - H$ depends on c. Therefore, the order of all the potential solution pieces in terms of x is parameter dependent. Moreover, while $\bar{v}(x)$ is expected to be concave the nonlocal term $\Delta\bar{v}(x)$ does not allow to guarantee this and therefore the a priori ordering of the pieces cannot be fully determined. Compared to the basic (2.13), (2.19) is of the form $(A_1 - v')^2 + (A_2 - v')^2 = rv$ so solving the quadratic for v' it is possible to obtain multiple positive roots. Determining the correct root is then done by enforcing the C^1 continuity of v', i.e. making sure that $x \mapsto v'(x)$ is continuous (and decreasing).

Lemma 2.2 illustrates what happens when the macroeconomic environment becomes more volatile. The parameter ϵ can be thought of as a proxy for volatility. The lemma shows the homogenization arising as $\epsilon \to 0$. Indeed, increasing ϵ can be viewed as increase in market volatility. Figure 2.3 illustrates the behavior of the value functions and controls in terms of ϵ. The value and marginal value of reserves decreases in high regime and increases in low regime. Therefore the production rate increases in high regime and decreases in low regime, since holding reserves becomes more valuable in high regime and less valuable in low regime. Similarly, as $\epsilon \searrow 0$, the exploration effort decreases in high regime due to decreased marginal value of a new discovery and increases in low regime.

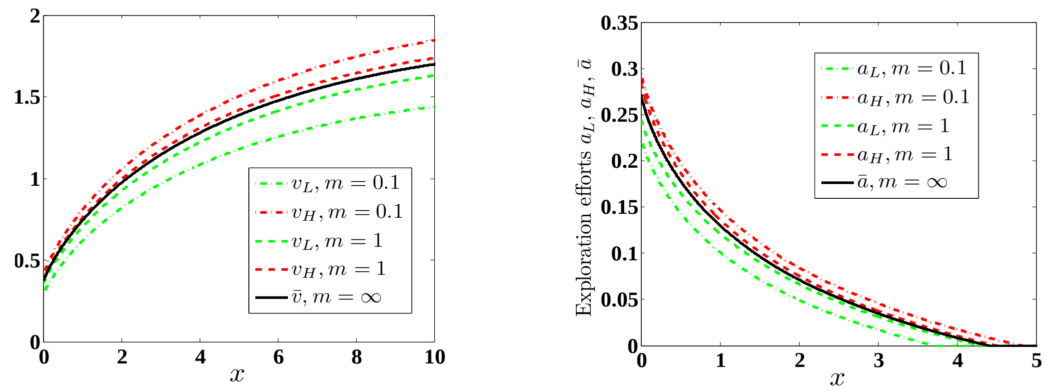

Figure 2.3: Left panel: convergence of the game functions $v_i(x) \to \bar{v}(x)$ as $\lambda_{01}, \lambda_{10} \to \infty$ together. Right panel: convergence of the exploration effort $a_i(x) \to \bar{a}(x)$. We take $\lambda_{01} = m/3, \lambda_{10} = m/5$, with $m = 0.1, 1$ as well as the limiting solution defined in Lemma 2.2.

2.3.2 Production Shut-Down in Low-Demand Regime

Due to fluctuating profit levels across the macroeconomic regimes, there may arise situations in which player 1 voluntarily shuts down production in the low demand regime when reserves level x is small. We define $x_{start} := \inf\{x > 0 : q_L(x) > 0\}$, the critical reserves level below which production stops. Heuristically, below x_{start} marginal value of reserves is so high relative to the low price offered that $v'_L(x) > L$ leading to $q^1_L(x) = 0$ in (2.7).

As mentioned before, such a shutdown (M2-equilibrium) can only happen in the low regime and is driven by the expectation of collecting higher revenue once demand reverts to the high regime. Typically, x_{start} is quite low, so M2-equilibrium arises just before total exhaustion of reserves. There are two motivations for production shut-down. The *quantity effect* reflects the extra profits available under high demand. Thus, a sufficiently large difference in demand levels across regimes is required for an M2 equilibrium to appear. The *time effect* reflects the anticipated waiting time until high demand which is important due to the present value discounting involved. Thus, for M2 equilibrium it is necessary that the holding time in the low regime is sufficiently short relative to the discount rate r.

Figure 2.4 illustrates these phenomena. The left panel of Figure 2.4 shows that x_{start} is increasing in λ_{01}. This is the time effect: as λ_{01} increases, the exhaustible producer is anticipating imminent higher profits and is more willing to shut down production to save reserves for that purpose. Asymptotically, as $\lambda_{01} \to \infty$, x_{start} converges to $\tilde{x}_{start} := \inf\{x > 0 : L - \tilde{v}_H(x) > 0\}$, where $\tilde{v}_H(x)$ is the game function corresponding to the constant-high-demand case, see Lemma 2.1. The middle panel of Figure 2.4 shows that when H is close to L there is no production shutdown, whereas as H increases, x_{start} increases in H unboundedly. This is the quantity effect: larger H increases the marginal value of reserves which makes the situation $v'_L(x) > L$ more likely.

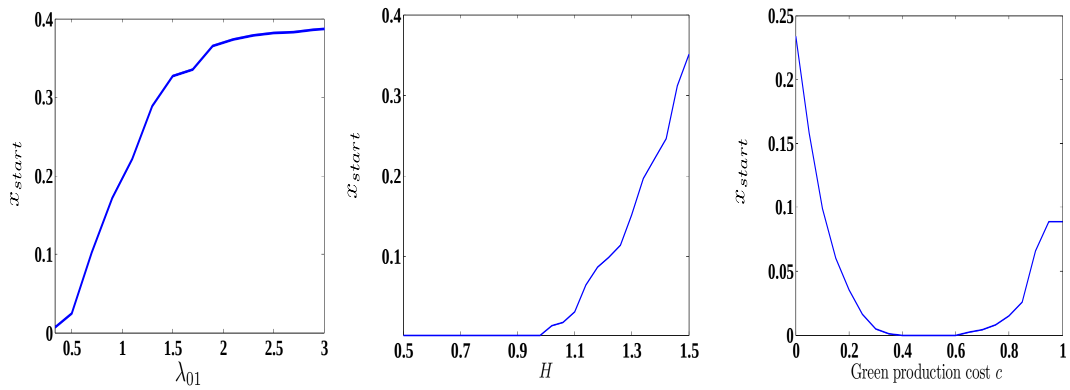

Figure 2.4: Left panel: x_{start} as a function of $\lambda_{01} \in [1/3, 3]$. Middle panel: x_{start} as a function of $H \in [L, 1.5]$. Right panel: x_{start} as a function of green production cost $c \in [0, H]$. The default parameters are $L = 0.5, H = 1, c = 0.3, \lambda_{01} = \frac{1}{3}, \lambda_{10} = \frac{1}{5}$.

Finally, the right panel of Figure 2.4 shows x_{start} as a function of green production cost c. We observe an ambiguous effect of competition on voluntary shutdown. Because shutdown only happens with very low reserves, it takes place when the green producer 2 is generally the "leader" of the market and hence the equilibrium rates are very sensitive to the green leader's costs. When c is very small, competition lowers the value of reserves and makes the marginal value large. Therefore, x_{start} is large when c is small. At moderate c, the competition is alleviated and the game becomes in favor of the exhaustible producer 1, thus the exhaustible production is expanded due to decreased marginal value of reserves. When c is close to L, green production is very low or even blockaded in low regime, causing producer 1 to *raise* production under low demand, and eschew shutdown $x_{start} = 0$. As c increases beyond $c > L$, the exhaustible producer begins to lead the market under both regimes, and is driven by the quantity/time effects to shutdown production. Thus, x_{start} increases in c when $L < c < H$. When $c = H$, the green producer 2 is completely blockaded in both the low and high regimes, thus the model reduces to exhaustible production monopoly. It also deserves our attention that in the setting $c \approx 0$ that is most favorable to the green producer, x_{start} is significantly larger than in a player 1 monopoly market ($c = H$). This is because competition encourages producer 1 to conserve reserves in low regime, much more so than in the monopoly setting.

2.4 Extensions of Basic Model

2.4.1 Multiple Demand Regimes

We could similarly analyze the situation where the market demand switches among $n > 2$ macroeconomics regimes such that the demand function is given by

$$p_t = M_t - q^1 - q^2, \quad M_t \in \{D_1, D_2, ..., D_n\}, \quad 0 < D_1 < D_2 < ... < D_n.$$

Denote the generator of the Markov chain (M_t) as

$$\Lambda = \begin{array}{cc} & \begin{array}{cccc} D_1 & D_2 & \cdots & D_n \end{array} \\ \begin{array}{c} D_1 \\ D_2 \\ \vdots \\ D_n \end{array} & \begin{pmatrix} -\sum_{j\neq 1}\lambda_{1j} & \lambda_{12} & \cdots & \lambda_{1n} \\ \lambda_{21} & -\sum_{j\neq 2}\lambda_{2j} & \cdots & \lambda_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{n1} & \lambda_{n2} & \cdots & -\sum_{j\neq n}\lambda_{nj} \end{pmatrix} \end{array}$$

Then (2.3) is generalized to

$$\sup_{q_i^1}\left[q_i^1(x)\left(D_i - q_i^1(x) - q_i^{2,*}(x)\right) - v_i'(x)q_i^1(x)\right] + \sup_{a_i}\left[a_i\lambda\Delta v_i(x) - C_a(a_i)\right]$$

$$+\sum_{j\neq i}\lambda_{ij}(v_j(x) - v_i(x)) - rv_i(x) = 0, \qquad (2.21)$$

and similarly for $g_i(x)$, and the boundary conditions are

$$v_i(0) = \sup_{a_i\geq 0}\frac{\sum_{j\neq i}v_j(0)\lambda_{ij} + v_i(\delta)\lambda a_i - C_a(a_i)}{r + \sum_{j\neq i}\lambda_{ij} + \lambda a_i}, \quad i = 1, 2, ..., n. \qquad (2.22)$$

The candidate optimizers $a_i^*(x), q_i^{\ell,*}(x)$, $i = L, H$, $l = 1, 2$, remains as in (2.7)-(2.8).

With multiple regimes, some of the intuitive comparative statics become unavailable. For example, Conjecture 1 states that with only two regimes the production rates (and hence the shadow marginal costs of reserves) appear to be always ordered. Namely, production and exploration efforts are larger under higher demand. For a generic chain (M_t), such monotonicity no longer necessarily holds.

As explained, the game values can be thought of as "averages" of the corresponding rewards under fixed demand regimes. The averaging is done in terms of the expected discounted time spent in each regime given M_0. With more than two regimes, this averaging is non-trivial: even if M_0 is high today, the future prospects could be worse compared to a lower starting point. For instance, consider the case where the market demand switches cyclically among three levels $D_1 < D_2 < D_3$, with a generator of the form

$$\Lambda = \begin{pmatrix} -\lambda_{12} & \lambda_{12} & 0 \\ 0 & -\lambda_{23} & \lambda_{23} \\ \lambda_{31} & 0 & -\lambda_{31} \end{pmatrix}. \tag{2.23}$$

Thus, (M_t) moves cyclically along $D_1 \to D_2 \to D_3 \to D_1$. If the proportion of time spent in regime 2 is significantly longer than in regime 3 (namely $\lambda_{12} \ll \lambda_{23} \ll \lambda_{31}$), then regime 3 could be a worse starting point than regime 2. This situation is illustrated in Figure 2.5 that shows that $v_i(x)$ are no longer monotone (and neither are $q_i^1(x)$ or $a_i(x)$) in i. Therefore, the original ordering of D_i gets shuffled due to the influence of the transition rates λ_{ij}.

A further generalization would be to consider continuous fluctuations of market demand, for instance taking (M_t) as an Ito diffusion modulating the price function. Some of the standard choices could include a (Geometric) Brownian motion factor to model for example the evolution of total economy GDP, or a

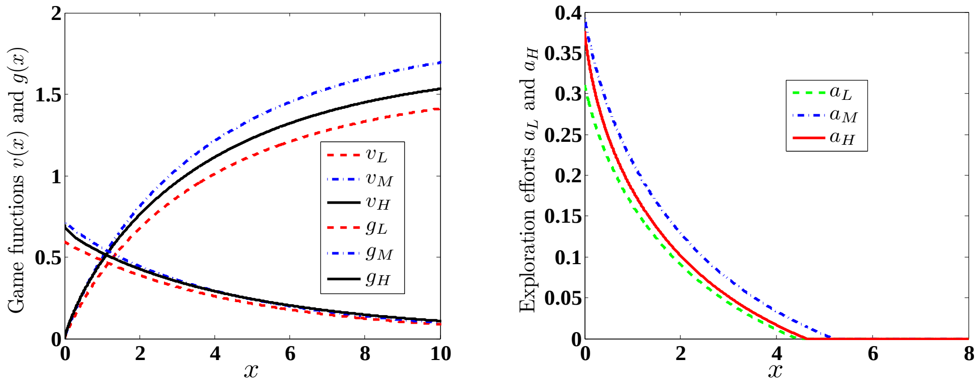

Figure 2.5: Duopoly with three demand regimes, $D_1 = L, D_2 = M, D_3 = H$. Left panel: Game functions $v_i(x)$ and $g_i(x)$, $i = L, M, H$ of the two producers. Right panel: exploration efforts $a_i(x)$, $i = L, M, H$. We take $L = 0.5, M = 1, H = 1.5$ and $\lambda_{12} = \frac{1}{2}, \lambda_{23} = \frac{1}{4}, \lambda_{31} = 1$ in (2.23).

stationary Ornstein-Uhlenbeck factor to model the macroeconomic business cycle relative to the long-run baseline. Such a model would require replacing the current HJB-I equations (2.3)-(2.4) by a partial differential equation since the new game values would be a function $v(x, m)$ of both the reserves level x and the demand level $M_0 = m$. If M_t is a diffusion, this will lead to a parabolic HJB-I system. The present theory of such systems, including smoothness of the game values, existence of equilibrium, etc. is not well-developed. In our context some of the additional difficulties include (i) non-standard implicit boundary conditions at $x = 0$; (ii) non-local terms arising from discrete exploration discoveries; (iii) degeneracy in the x variable that has only first-order dynamics (no volatility) and is fully endogenized by the player strategies. Overcoming all these challenges (that extend beyond the analytic properties to numerics and economic interpretation) is well beyond the scope of this paper. Indeed, we believe that the extra realism gained from incorporating (M_t) as a continuous factor is not worth the additional model complexity. Our take is that all models are stylized and are aiming at basic insights rather than complete practicality.

2.4.2 Stochastic Production Costs

One may use the stochastic factor to modulate other game parameters. For example, the production costs of player 2 (the green producer) might be changing over time. Such fluctuations could be due to varying technology costs; changing government policies such as renewable energy subsidies; or non-constant financing costs. To capture this setup we could then assume that $c = c(M_t)$ is modulated by the chain (M_t) whereas the demand is for simplicity now fixed at some $\bar{D}$.

The resulting game values would solve HJB-I equations essentially matching (2.3)-(2.4), except that player 2 production costs $c_i = c(D_i)$ now differ across regimes. Figure 2.6 illustrates the solution assuming a two-state (M_t). The parameters are similar to those in Figure 2.2 allowing a degree of comparison. As before, green production rises when costs are low (regime 1) and falls when costs become larger. In particular, the game can switch between Type I and type M1 equilibria due to regime change. Moreover, because exploration efforts of player 1 are not monotone in c in the setup of Section 2.2, they are also non-monotone here, see right panel of Figure 2.6. Thus, for some reserve levels x, a drop in competitor's production costs may induce increased motivation to explore.

Remark 2.4. *One could also imagine fluctuating production costs of player 1. In fact, with linear inverse demand, one may interpret the choke level D in $p = D - q^1 - q^2$ as the* net *difference between demand level and production costs for the exhaustible producer (recall that in Section 2.2 we took these fixed production costs to be zero for convenience). Hence, the original model that takes $D = D(M_t)$ is equivalent to assuming that $c^1 = c^1(M_t)$ with a baseline case $c^1(D_1) = 0$.*

One could also modulate other parts of the model, such as the exploration costs $C(a; M_t)$ (indeed, there was plenty of evidence that E&P costs in the oil industry rose sharply during the bull oil market of 2006-08 as increased demand spurred all companies to replenish reserves). Another idea that was suggested in [22] is to modulate the discovery rate λ (so as to capture for instance approach of the global exhaustibility of the resource). In general, one could mix and match above features, taking both $c(M_t)$ and $D(M_t)$ to be dependent on the macroeconomy. In that situation, there is no longer a clear ordering of the regimes. For example, it could be the case that in regime 1, demand is high but so is the competition with

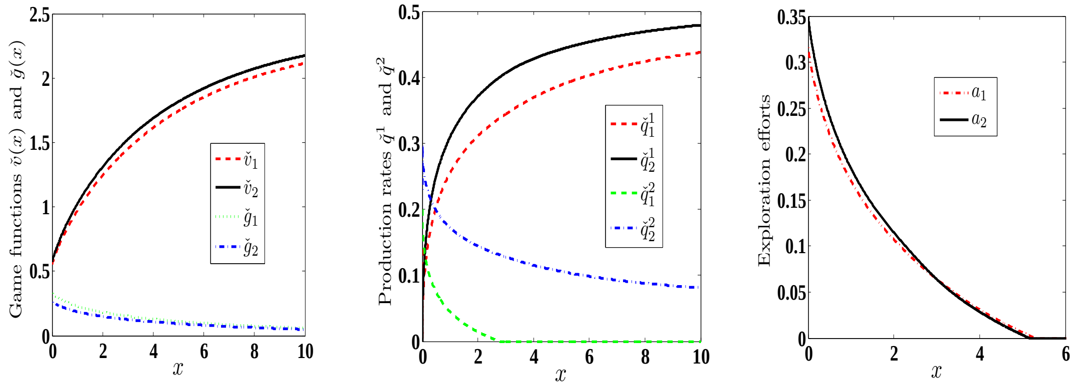

Figure 2.6: Duopoly with regime-switching green production costs. Left panel: Game functions $\check{v}_i(x)$ and $\check{g}_i(x)$, $i = 1, 2$ of the two producers. Middle panel: Production rates $q_i^\ell(x)$ of the two producers $\ell = 1, 2$. Right pane: Exploration efforts $a_i(x)$ of the exhaustible producer. We take $\bar{D} = 1, c_1 = 0.4, c_2 = 0.6$, and switching rates $\lambda_{01} = \frac{1}{3}, \lambda_{10} = \frac{1}{5}$. Exploration costs are $C_a(a) = 0.1a + a^2/2$.

green production; in regime 1 demand is lower but green producer is blockaded. Our above investigations can serve as a guide to disentangle the opposite effects that would then be induced on $q^\ell(x)$ and $a(x)$.

2.4.3 More Players

Our main model in this chapter featured two players. These are meant to be representative of the exhaustible and renewable producers, for example oil and "green" energy industries competing on the electricity market. By having only a single player with reserves, our continuous state variable X_t is one-dimensional which greatly facilitates the analysis. It would of course be more realistic to include multiple players with reserves (e.g. different type of conventional fossil fuels). An immediate extension would be to analyze the duopoly among two exhaustible producers. The difference in their reserves X_t^1, X_t^2 would determine the asymmetry in the game and decide who is the leader based on the respective shadow reserve costs. A version of such a model without exploration or stochastic demand was treated in [31]. Given the major challenges encountered there (in particular regularity issues for the game functions), this is another extension that we are not able to fully address here.

Beyond two players, the theory of Cournot oligopolies is rudimentary. See [39] for analysis of Bertrand oligopolies in deterministic markets (with no exploration nor stochastic factor). Consideration of more than two non-symmetric players will necessarily be challenging due to the exploding dimensionality. Moreover, it raises thorny questions regarding private and public information, namely whether all players can be fully informed about all other players given the complexity of the game. Since real life markets actually feature hundreds of agents, another useful approximation would be to study an infinity of players using the framework of mean field games [4]. Namely, one might consider the strategic interaction among a continuum of exhaustible producers with exploration. Related models without exploration was analyzed in the deterministic case by [29] and in the stochastic case (where reserves X_t receive Brownian shocks) by [16]. In the following chapters 3 and 4, we study the Cournot game with a continuum of players in the mean field game framework. What is new in our research, comparing with [29, 16], lies in that the exploration effort is modeled by a controlled jump process in the mean field game framework.

Chapter 3

Mean Field Games with a Continuum of Producers

In this chapter we study a specific mean field game model with a continuum of exhaustible producers in terms of both analytical and numerical aspects. This chapter is organized as follows. In Section 3.1, we introduce the MFG model in the limit $N \to \infty$ corresponding to the N-player Cournot game model introduced in section 1.3.4. In Section 3.2, we discuss the doubly coupled system of HJB and transport equations that characterize the mean field game Nash equilibrium. In Section 3.3, we introduce numerical methods to solve the system of HJB and transport equations for the mean field game Nash equilibrium. Some numerical examples are presented.

3.1 Mean field game problem with a continuum of players

As number of players becomes very large $N \to \infty$, thanks to the Law of Large Numbers, the empirical distribution η^N, defined by (1.8), is expected to converge to the normalized CDF η. The function $\eta(t, \cdot)$ is regarded as the reserves distribution among all players at date t, which means, for a given t and x, the proportion of all players at time t with reserves level greater than or equal to x.

Correspondingly the production and exploration controls (1.9) take the Markovian feedback form

$$q_t = q(t, X_t; \eta(t, \cdot)), \quad a_t = a(t, X_t; \eta(t, \cdot)). \tag{3.1}$$

To re-solve for the supply-demand equilibrium clearing price, we analyze the total averaged quantity $Q(t)$ of production at time t, defined as the (Stieltjes) integral of a representative player's quantity of production with respect to the reserves distribution,

$$Q(t) := -\int_0^\infty q(t, x; \eta)\ \eta(t, dx). \tag{3.2}$$

Note that $\eta(t, x)$ is decreasing in x, thus we add a negative sign to the integral in order to keep $Q(t)$ positive. In turn, $Q(t)$ determines the market price via the inverse demand function

$$p(t) = D^{-1}(Q(t)) = L + \int_0^\infty q(t,x;\eta)\ \eta(t,dx), \tag{3.3}$$

where the parameter L can be regarded as the cap on prices as supply vanishes.

For a representative player who starts with initial reserves level $X_t = x$, the mean-field objective functional is defined analogously to (1.5):

$$\mathcal{J}(q,a;t,x,\eta) := \mathbb{E}\left\{ \int_t^T \left[D^{-1}(Q(s))\, q_s - C_q(q_s) - C_a(a_s)\right] e^{-r(s-t)}\ ds \,\middle|\, X_t = x, \eta \right\}. \tag{3.4}$$

Above the strategies (q_t, a_t) take the Markovian feedback form (3.1) and the reserves distribution $(\eta(t,\cdot))$ is a probability CDF for all $s \in [t,T]$. We again remark that the profit of a player depends on all the other players through the mean field term $Q(t)$. According to [3], we define the mean field game Nash equilibrium of our model as

Definition 3.1 (Mean field game Markov Nash equilibrium)**.** *A MFG MNE is a triple (q^*, a^*, η^*) of processes on $[0,T]$ such that, denoting by X_t^* the solution of*

$$dX_t^* = -q_t^* dt + \delta dN_t^*, \quad t \geq 0, \qquad X_0^* \sim \eta^*(0,\cdot), \tag{3.5}$$

then $\eta^*(t,\cdot) = \mathbb{P}(X_t^* \geq x)$ is the distribution of X_t^*, $\forall t \in [0,T]$ and

$$\mathcal{J}(q^*, a^*; t, x, \eta^*) \geq \mathcal{J}(q, a; t, x, \eta^*),\ \forall (q,a) \in \mathcal{A}. \tag{3.6}$$

Remark 3.1. *The definition of MFG equilibrium 3.1 consists of two conditions. One condition, which we can call optimality condition, is that each producer chooses strategy* (q^*, a^*) *which gives optimal game value, given the others' strategies. The second condition, which we can call consistency condition, is that the reserves dynamics of each player under the control of the strategy* (q^*, a^*) *has the upper cumulative distribution function* η^* *that is the same as the one that enters the objective functional.*

In Section 3.2, we introduce a numerical scheme to search for the mean field game Nash equilibrium defined in Definition 3.1, which is the core problem of this paper.

3.2 Mean field game Nash equilibrium

According to [34, 38], solving for MFG MNE involves two partial differential equations. One equation is the HJB equation of the game value function of a representative player, which is derived by a dynamic programming principle. Optimal production and exploration strategies (q^*, a^*) can be obtained from the HJB equation. The other equation is the transport equation for the distribution η^* of reserves process X^* controlled by the strategies (q^*, a^*) obtained from the HJB equation.

Section 3.2.1, treats the HJB equation associated to the game value function of a representative agent. The PDE that characterizes the evolution of the reserves distribution will be discussed in Section 3.2.2. The overall coupled system associated to the MFG MNE is taken up in Section 3.2.3 and approached through an iterative scheme similar to [29, 16]. Details of numerical methods and examples will be discussed in Section 3.3.

3.2.1 Game value function of a representative player

Let us fix a sequence of probability CDF's $\eta(t,\cdot)$. Associated with the objective functional (3.4), we define the game value function $v^\eta(t,x)$ of a representative player by

$$\begin{aligned} v^\eta(t,x) &:= \sup_{(q,a)\in\mathcal{A}} \mathcal{J}\left(q,a;t,x,\eta\right) \\ &= \sup_{(q,a)\in\mathcal{A}} \mathbb{E}\left\{ \int_t^T \left[p(s;\eta)q_s - C_q(q_s) - C_a(a_s)\right] e^{-r(s-t)} ds \;\middle|\; X_t = x \right\}, \end{aligned} \tag{3.7}$$

where the player chooses optimal production rate $q(t,X_t;\eta)$ and exploration rate $a(t,X_t;\eta)$ from the set $\mathcal{A}$ of Markovian feedback controls. Note that above η is treated as an exogenous parameter, while the price $p(\cdot;\eta)$ is still endogenous being a function of total production $p(t;\eta(t,\cdot)) = D^{-1}\left(Q(t)\right)$. As we will see, this in fact introduces a global dependence between the map $x \mapsto q(t,x)$ and $p(t)$.

Define the forward difference operator Δ_x as $\Delta_x v(t,x) := v(t,x+\delta) - v(t,x)$.

Lemma 3.1. *The game value function $v(t,x)$ defined by* (3.7) *satisfies the following HJB equation*

$$0 = \frac{\partial}{\partial t} v^\eta(t,x) - r v^\eta(t,x) + \frac{1}{2\beta_1}\left[\left(p(t;\eta(t,\cdot)) - \kappa_1 - \frac{\partial}{\partial x} v^\eta(t,x)\right)^+\right]^2$$
$$+ \frac{1}{2\beta_2}\left[(\lambda(t)\Delta_x v^\eta(t,x) - \kappa_2)^+\right]^2, \tag{3.8}$$

with terminal condition $v^\eta(T,x) = 0$, where the optimal $q^\eta(t,x)$ and $a^\eta(t,x)$ are given by

$$q^\eta(t,x) = \frac{1}{\beta_1}\left(L - Q(t) - \kappa_1 - \frac{\partial}{\partial x} v^\eta(t,x)\right)^+, \tag{3.9}$$
$$a^\eta(t,x) = \frac{1}{\beta_2}\left(\lambda(t)\Delta_x v^\eta(t,x) - \kappa_2\right)^+, \tag{3.10}$$

with $Q^\eta(t)$ uniquely determined by the equation

$$Q^\eta(t) + \int_0^\infty \frac{1}{\beta_1}\left(L - \kappa_1 - \frac{\partial}{\partial x} v^\eta(t,x) - Q^\eta(t)\right)^+ \eta(t,dx) = 0. \tag{3.11}$$

The price $p(t)$ depends on q^η and the given reserves distribution η via (3.3)*.*

Proof. The associated HJB equation of (3.7) derived by the dynamic programming principle, is

$$0 = \frac{\partial}{\partial t} v^\eta(t,x) - r v^\eta(t,x) + \sup_{a\geq 0}\left[-C_a(a) + a\lambda(t)\Delta_x v^\eta(t,x)\right] \tag{3.12}$$
$$+ \sup_{q\geq 0}\left[p(t;\eta(t,\cdot))q - C_q(q) - q\frac{\partial}{\partial x} v^\eta(t,x)\right],$$

where the forward difference term $\Delta_x v(t,x)$ is due to the jumps in the reserves dynamics. The optimal exploration rate a^η is determined by the first order condition

$$a^\eta(t,x) = \arg\max_{a\geq 0}\left[-C_a(a) + a\lambda(t)\Delta_x v^\eta(t,x)\right] = \frac{1}{\beta_2}\left(\lambda(t)\Delta_x v^\eta(t,x) - \kappa_2\right)^+, \tag{3.13}$$

where we plugged the quadratic form of C_a from (1.3). Similarly, the optimal production rate q^η should satisfy

$$q^\eta(t,x) = \mathrm{argmax}_{q\geq 0}\left[p(t,\eta(t,\cdot))q - C_q(q)\right].$$

The first order condition for $q^\eta(t,x)$ is

$$\begin{aligned} 0 &= \frac{\partial}{\partial q}\left[p(t,\eta(t,\cdot))q^*(t,x) - C_q(q^\eta(t,x)) - q^\eta(t,x)\frac{\partial}{\partial x}v^\eta(t,x)\right] \\ \Leftrightarrow \quad \beta_1 q^\eta(t,x) &= \left(p(t,\eta(t,\cdot)) - \kappa_1 - \frac{\partial}{\partial x}v^\eta(t,x)\right)^+. \end{aligned} \tag{3.14}$$

Recalling that $p(t,\eta(t,\cdot)) = L - Q^\eta(t)$ yields (3.9). Integrating the right-hand side of (3.9) with respect to $\eta(t,\cdot)$,

$$\int_0^\infty \frac{1}{\beta_1}\left(L - \kappa_1 - \frac{\partial}{\partial x}v^\eta(t,x) - Q(t)\right)^+ \eta(t,dx) = \int_0^\infty q^\eta(t,x)\eta(t,dx) = -Q(t). \tag{3.15}$$

Thus, $Q^\eta(t))$ satisfies $G(Q^\eta(t)) = 0$ as in (3.11) where

$$G(Q) = Q + \int_0^\infty \frac{1}{\beta_1}\left(L - \kappa_1 - \frac{\partial}{\partial x}v^\eta(t,x) - Q\right)^+ \eta(t,dx).$$

Assuming $L > \kappa_1$ (otherwise production is never profitable and $Q(t) = 0$), we have $G(0) > 0$ and $G((L-\kappa_1)^+) < 0$, and a unique root $Q(t)$ exists $\in [0, L-\kappa_1]$ since $Q \mapsto G(Q)$ is continuous and strictly decreasing. Note that $Q \mapsto G(Q)$ is continuous because for arbitrary $\epsilon \lesseqgtr 0$ we have as $\epsilon \to 0$ that

$$\begin{aligned}
\lim_{\epsilon\to 0} G(Q+\epsilon) &= \lim_{\epsilon\to 0}\left[Q+\epsilon+\int_0^\infty \frac{1}{\beta_1}\left(L-\kappa_1-\frac{\partial}{\partial x}v^\eta(t,x)-Q-\epsilon\right)^+\eta(t,dx)\right] \\
&= Q+\lim_{\epsilon\to 0}\int_0^\infty \frac{1}{\beta_1}\left(L-\kappa_1-\frac{\partial}{\partial x}v^\eta(t,x)-Q-\epsilon\right)^+\eta(t,dx) \\
&= Q+\int_0^\infty \lim_{\epsilon\to 0}\frac{1}{\beta_1}\left(L-\kappa_1-\frac{\partial}{\partial x}v^\eta(t,x)-Q-\epsilon\right)^+\eta(t,dx) \\
&= Q+\int_0^\infty \frac{1}{\beta_1}\left(L-\kappa_1-\frac{\partial}{\partial x}v^\eta(t,x)-Q\right)^+\eta(t,dx) \\
&= G(Q),
\end{aligned}$$

where the exchange of the limit $\lim_{\epsilon\to 0}$ and the integral $\int_0^\infty \cdot\, \eta(t,dx)$ is justified by bounded convergence theorem, as $-\int_0^\infty 1\ \eta(t,dx) = 1 < \infty$ and the integrand is uniformly bounded for all ϵ, i.e., $\left|\frac{1}{\beta_1}\left(L-\kappa_1-\frac{\partial}{\partial x}v^\eta(t,x)-Q-\epsilon\right)^+\right| \le \frac{1}{\beta_1}(L-\kappa_1)$. Finally (3.8) follows by using (3.13) and (3.14) in (3.12).

□

We observe two non-standard features of the HJB equation (3.8). First, the optimal production control (3.9) does not only depend on the individual producer's value function $\frac{\partial}{\partial x}v^\eta(t,x)$, but also on the reserves distribution of all the players through the mean field term $\int_0^\infty \frac{\partial}{\partial z}v^\eta(t,z)\eta(t,dz)$. Second, (3.8) contains two non-local terms: the forward difference $v^\eta(t, x + \delta) - v^\eta(t, x)$ and the integral $\int_0^\infty \frac{\partial}{\partial z} v^\eta(t, z)\eta(t, dz)$.

The HJB equation has two boundary conditions. First, at $t = T$ we take $v(T, x) = 0$ as no more production is assumed possible beyond the prescribed horizon. Second, the exhaustibility condition $x \geq 0$ imposes a boundary condition at $x = 0$ similar to the model [40] for a single exhaustible producer. Since production $q(t, 0) = 0$ is zero on the boundary $x = 0$, the game value function $v^\eta(t, x)|_{x=0}$ satisfies

$$\begin{aligned} 0 &= \frac{\partial}{\partial t} v^\eta(t, 0) - r v^\eta(t, 0) + \sup_{a \geq 0} \left[-C_a(a) + a\lambda(t)\Delta_x v^\eta(t, 0) \right] \\ &= \frac{\partial}{\partial t} v^\eta(t, 0) - r v^\eta(t, 0) + \frac{1}{2\beta_2} \left[(\lambda(t)\Delta_x v^\eta(t, 0) - \kappa_2)^+ \right]^2, \qquad 0 \leq t < T. \end{aligned} \tag{3.16}$$

We will use the boundary condition equation (3.16) in the numerical schemes.

3.2.2 Transport equation of reserves distribution

In this section we study evolution of the reserves distribution through the transport equation of the upper-cumulative distribution function $\eta(t, \cdot)$ of the reserves process X_t from (1.2) where N_t is a point process with controlled rate $\lambda(t)a_t$, and the production rate $q_t = q(t, X_t)$ and exploration rate $a_t = a(t, X_t)$ are given, i.e. treated as exogenous inputs.

When reserves reach zero $X_t = 0$, production shuts down $q_t = 0$. With exploration effort being made, the reserves level X_t can bounce back to $X_\tau = \delta$, however the waiting time until next discovery is strictly positive. As a result, $\mathbb{P}(X_t = 0) > 0$, i.e. the distribution of X_t has a point mass at $x = 0$. Thus to study the evolution of the distribution of X_t, we consider two parts: the upper-cumulative distribution function $\eta(t,x) = \mathbb{P}(X_t \geq x)$ in the interior $x > 0$; and the boundary probability $\pi(t) := \mathbb{P}(X_t = 0) = 1 - \eta(t, 0+)$. The upper-CDF $\eta(t,x)$ is regarded as the proportion of players with reserves level greater than or equal to x, and $\pi(t)$ is interpreted as the proportion of producers with no reserves. The following proposition gives the system of PDEs that the pair $(\pi(t), \eta(t,x))$ satisfy. See the proof in Appendix A.3.

Proposition 3.1 (Transport equation)**.** *The distribution of the reserves process X_t is characterized by the pair $(\pi(t), \eta(t,x))$, where $\pi(t) = \mathbb{P}(X_t = 0)$, $\eta(t,x) = \mathbb{P}(X_t \geq x)$, $0 < x < \infty$, satisfy the following system of differential equations (note that the partial x-derivative on the first line is taken from the right $x \downarrow 0$ since*

$1 = \eta(t,0) > \eta(t,0+)$ *is discontinuous at* $x = 0$*)*

$$\pi(t) = 1 - \eta(t,0+); \tag{3.17a}$$

$$\frac{\partial}{\partial t}\eta(t,x) = \lambda(t)a(t,0)\pi(t) - \int_{0+}^{x} \lambda(t)a(t,z)\eta(t,dz) + q(t,x)\frac{\partial}{\partial x}\eta(t,x), \ 0 < x \le \delta; \tag{3.17b}$$

$$\frac{\partial}{\partial t}\eta(t,x) = -\int_{x-\delta}^{x} \lambda(t)a(t,z)\eta(t,dz) + q(t,x)\frac{\partial}{\partial x}\eta(t,x), \quad x > \delta. \tag{3.17c}$$

with given initial condition $\eta(0,x) = \eta_0(x)$ *and* $\pi(0) = p_0 = 1 - \eta_0(0+)$.

The discontinuity of $\eta(t,\cdot)$ at $x = 0$ generates higher order discontinuities at $x = \delta, 2\delta, 3\delta, \cdots$. Indeed, at $x = k\delta$ only the first $(k-1)$ derivatives of $\eta(t,x)$ exist. In other words, the distribution of X_t has a point mass at $x = 0$, a first-order discontinuity (non-continuous density) at $x = \delta$ and a smooth density for all other $x > 0$. This non-smoothness is the reason why we do not work with the density $m(t,x) = -\frac{\partial}{\partial x}\eta(t,x)$.

Remark 3.2. *The size of* δ *of each new discovery can be random in general. We may model discovery amounts via a stochastic sequence* $\delta_n, n = 1, 2, \ldots,$ *where each* δ_n *is identically distributed with some distribution* $F_\delta(\cdot)$ *and independent of everything else in the model. Introducing* F_δ *entails replacing the integral* $\int_{x-\delta}^{x} \lambda(t)a(t,z)\eta(t,dz)$ *in* (3.17c) *with* $\int_0^\infty F(du)\int_{x-u}^{x} \lambda(t)a(t,z)\eta(t,dz)$*. Similarly, in the HJB equation we would replace* $v(t,x+\delta)$ *with* $\int_0^\infty v(t,x+u)F_\delta(du)$*. For simplicity we stick to fixed discovery sizes for the rest of the article.*

3.2.3 System of HJB-transport equations

To recap, the MFG MNE is defined by the HJB equation (3.7) where we plug-in the equilibrium CDF η^* (and exhausted proportion $\pi^*(t)$), and the transport equation (3.17) where we plug-in the equilibrium q^* and a^*. The equilibrium price process is $p^*(t) = L + \int_0^\infty q^*(t,z)\eta^*(t,dz)$. The resulting system is summarized in the following.

Proposition 3.2 (MFG PDEs)**.** *The mean field game Nash equilibrium* (q^*, a^*, η^*) *is determined by the HJB equation:*

$$\begin{aligned} 0 = &\frac{\partial}{\partial t}v(t,x) - rv(t,x) + [-C_a(a^*(t,x)) + a^*(t,x)\lambda(t)\Delta_x v(t,x)] \\ &+ \left[p^*(t)q^*(t,x) - C_q(q^*(t,x)) - q^*(t,x)\frac{\partial}{\partial x}v(t,x)\right], \quad 0 < x,\ 0 \le t < T, \end{aligned} \tag{3.18}$$

where the $q^*(t,x)$ *and* $a^*(t,x)$ *are given by*

$$q^*(t,x) = \frac{1}{\beta_1}\left(L - Q(t) - \kappa_1 - \frac{\partial}{\partial x}v(t,x)\right)^+, \tag{3.19}$$

$$a^*(t,x) = \frac{1}{\beta_2}\left(\lambda(t)\Delta_x v(t,x) - \kappa_2\right)^+, \tag{3.20}$$

with $Q(t)$ *uniquely determined by the equation*

$$Q(t) = -\int_0^\infty \frac{1}{\beta_1}\left(L - \kappa_1 - \frac{\partial}{\partial x}v(t,x) - Q(t)\right)^+ \eta^*(t,dx),$$

and the transport equation:

$$\pi^*(t) = 1 - \eta^*(t, 0+); \tag{3.21a}$$

$$\frac{\partial}{\partial t}\eta^*(t,x) = \lambda(t)a^*(t,0)\pi^*(t) - \int_{0+}^{x} \lambda(t)a^*(t,z)\eta^*(t,dz) + q^*(t,x)\frac{\partial}{\partial x}\eta^*(t,x), \quad 0 < x \leq \delta; \tag{3.21b}$$

$$\frac{\partial}{\partial t}\eta^*(t,x) = -\int_{x-\delta}^{x} \lambda(t)a^*(t,z)\eta^*(t,dz) + q^*(t,x)\frac{\partial}{\partial x}\eta^*(t,x), \qquad x > \delta. \tag{3.21c}$$

The HJB equation and transport equation are doubly coupled with η^* entering the HJB equation through the aggregate production which is an integral of optimal production rates $q^*(t,x)$ with respect to the mean-field reserves distribution $\eta^*(t,dx)$. Conversely, the optimal production and exploration rates (q^*, a^*) obtained from the HJB equation of a representative player drive the reserves distribution η^*.

Existence, uniqueness, and regularity of the solutions of the system of MFG PDEs is still an ongoing challenge and an area of active research. For the system (3.18)- (3.21) the difficulty in proving existence and uniqueness of solutions lies in the non-local coupling term $\int_0^\infty \frac{\partial}{\partial x}v(t,x)\eta(t,dx)$ and forward delay term $\Delta_x v(t,x) = v(t,x+\delta) - v(t,x)$. Bensoussan et al. [4] gave a comprehensive study on linear-quadratic MFGs, in which the cost functional is quadratic in all state variables, control variables and the mean field terms; while the controlled dynamics of state variables are linear and also consist of mean field terms. It was shown that equilibrium solution exists uniquely for one dimension case. For dimension higher than one, a sufficient condition for the unique existence of the equilibrium strategy is also provided. Cardaliaguet et al. [6] gave detailed proof of existence and uniqueness of equilibrium solution for mean field games of first order with local coupling term. Cardaliaguet et al. [7] proved the existence and uniqueness of equilibrium solution for mean field game of second order with local coupling term. A local coupling term of a player at (t, x) is a function $F(t, x, m(t, x))$ that enters the integrand of game value function, where $m(t, x)$ is the density of players' states. The function $F(t, x, m(t, x))$ represents the interaction of a player at (t, x) with other players at the *same* location x and time t. In our model a player's game value $v(t, x)$ at time t and location x depends on all other players through the term $\int_0^\infty \frac{\partial}{\partial x} v(t, x)\eta(t, dx)$ which is non-local because it is the *sum* of all other players' marginal game values at the same time t.

A common feature that our model (3.18)-(3.21) shares with the Cournot models in [16, 17] is the non-local term $\int_0^\infty \frac{\partial}{\partial x} v(t, x)\eta(t, dx)$. Existence and uniqueness of the solutions to the system of second order mean field game partial differential equations introduced by [16, 17] are studied in [26, 27]. [26] proved existence and uniqueness of solutions to the system of mean field game partial differential equations in [16, 17] with Dirichlet boundary conditions that are associated with the situation where the reserves are limited and players are unable to renew their stock after exhaustion. [27] proved existence and uniqueness of solutions to the mean field game partial differential equations in [16, 17] with Neumann boundary conditions which are associated with the situation that players are able to renew their stock after exhaustion. [27] further proved existence and uniqueness of weak solutions to the corresponding first order system at the deterministic limit, which coincides with the non-exploration case in our model. That is, existence and uniqueness of solution to the system (3.18)-(3.21) with $\lambda = 0$ are proved, according to [27]. First order mean field game partial differential equations with forward delay term of the form $\Delta_x v(t,x) = v(t,x+\delta) - v(t,x)$ due to jump process in our model, to the best of our knowledge, has not been discussed in any literatures about existence, uniqueness, and regularity. The literatures [4, 6, 7, 26, 27] all consider mean field games with state variables driven by Brownian motions, which lead to second order partial differential equations.

The MFG framework links the individual strategic behavior of each producer with the macro-scale organization of the market. Therefore the main economic insights concern the resulting *aggregate* quantities that describe the overall evolution of the market. For this purpose, we let $Q(t)$ be the total production at time t, $A(t)$ the total discovery, and $R(t)$ the total reserves, which are defined respectively as

$$Q(t) = -\int_0^\infty q^*(t,x)\eta^*(t,dx), \tag{3.22}$$

$$R(t) = \int_0^\infty \eta^*(t,x)dx, \tag{3.23}$$

$$A(t) = -\delta \int_0^\infty \lambda(t)a^*(t,x)\eta^*(t,dx). \tag{3.24}$$

Note that $R(t) = \int_0^\infty \mathbb{P}(X_t \geq x)dx = \mathbb{E}[X_t]$ justifying its meaning of total reserves. The following Lemma 3.2 shows the relation between these quantities of interest. It can be interpreted as conservation of mass for the reserves: at the macro-scale total reserves change is simply the net difference between reserves additions (via new discoveries $A(\cdot)$) and reserves consumption (via production $Q(\cdot)$).

Lemma 3.2. *We have the relation*

$$\frac{d}{dt}R(t) = -Q(t) + A(t), \quad i.e. \quad R(t) = R(0) - \int_0^t Q(s)\ ds + \int_0^t A(s)\ ds. \tag{3.25}$$

Proof. We integrate both sides of the transport equation (3.21) with respect to x over $(0,\infty]$ to obtain

$$\begin{aligned}
&\int_{0+}^\infty \frac{\partial}{\partial t}\eta^*(t,x)dx \\
&= -\overbrace{\int_{0+}^\delta \left(\int_{0+}^x \lambda(t)a^*(t,z)\eta^*(t,dz)\right)\ dx}^{=:I1} - \overbrace{\int_\delta^\infty \left(\int_{x-\delta}^x \lambda(t)a^*(t,z)\eta^*(t,dz)\right)\ dx}^{=:I2} \\
&\quad + \int_{0+}^\infty \left(q^*(t,x)\frac{\partial}{\partial x}\eta^*(t,x)\right) dx.
\end{aligned} \tag{3.26}$$

For the last term by the definition of the Stieltjes integral, the integrator is equivalently $\frac{\partial}{\partial x}\eta^*(t,x)dx = \eta^*(t,dx)$. We apply integration-by-parts to the first two terms of the right hand side of (3.26) to obtain

$$\begin{aligned}
I1 &= \left[x\int_{0+}^{x}\lambda(t)a^*(t,z)\eta^*(t,dz)\right]_{0+}^{\delta} - \int_{0+}^{\delta} x\frac{\partial}{\partial x}\left(\int_{0+}^{x}\lambda(t)a^*(t,z)\eta^*(t,dz)\right)dx \\
&= \delta\int_{0+}^{\delta}\lambda(t)a^*(t,z)\eta^*(t,dz) - \int_{0+}^{\delta} x\lambda(t)a^*(t,x)\eta^*(t,dx), \qquad \text{and} \qquad (3.27) \\
I2 &= \left[x\int_{x-\delta}^{x}\lambda(t)a^*(t,z)\eta^*(t,dz)\right]_{x=\delta}^{x=\infty} - \int_{\delta}^{\infty} x\frac{\partial}{\partial x}\left(\int_{x-\delta}^{x}\lambda(t)a^*(t,z)\eta^*(t,dz)\right)dx \\
&= -\delta\int_{0+}^{\delta}\lambda(t)a^*(t,z)\eta^*(t,dz) \\
&\quad - \left[\int_{\delta}^{\infty} x\lambda(t)a^*(t,x)\eta^*(t,dx) - \int_{0+}^{\infty}(x+\delta)\lambda(t)a^*(t,x)\eta^*(t,dx)\right] \\
&= -\delta\int_{0+}^{\delta}\lambda(t)a^*(t,z)\eta^*(t,dz) + \int_{0+}^{\delta} x\lambda(t)a^*(t,x)\eta^*(t,dx) \\
&\quad + \delta\int_{0+}^{\infty}\lambda(t)a^*(t,x)\eta^*(t,dx). \qquad (3.28)
\end{aligned}$$

The left hand side of (3.26) can be written as

$$\int_{0+}^{\infty}\frac{\partial}{\partial t}\eta^*(t,x)dx = \frac{d}{dt}\int_{0+}^{\infty}\eta^*(t,x)dx, \qquad (3.29)$$

where the exchange of the partial differential operator and the integral is justified by the Leibniz integral rule under the condition that both $\eta^*(t,x)$ and $\frac{\partial}{\partial t}\eta^*(t,x)$ are continuous in the domain $(t,x)\in[0,\infty)\times(0,\infty)$. By substituting (3.27)-(3.29) into the equation (3.26), we have

$$\frac{d}{dt}\int_{0+}^{\infty}\eta^*(t,x)dx = -\delta\int_{0+}^{\infty}\lambda(t)a^*(t,x)\eta^*(t,dx) + \int_{0+}^{\infty}q^*(t,x)\eta^*(t,dx),$$

which gives (3.25). □

3.3 Numerical methods and examples

We use an iterative scheme to numerically solve the system of HJB equation (3.18) and transport equation (3.21), similar to the approach in [29, 16]. The Picard-like iterations start with an initial price process $p^{(0)}(t)$ as an input into the MFG value function (3.8), which reduces to a standard optimization problem for the production and exploration rates $(q^{(0)}, a^{(0)})$. Then we input $(q^{(0)}, a^{(0)})$ into the equation (3.17) of reserves evolution to solve for $\eta^{(0)}(\cdot, \cdot)$. The $q^{(0)}$ and $\eta^{(0)}$ obtained are used to update the price (3.3), i.e., $p^{(1)}(t) = \frac{1}{2}\left[D^{-1}\left(-\int_0^\infty q^{(0)}(t,x)\eta^{(0)}(t,dx)\right) + p^{(0)}\right]$. The updated price $p^{(1)}(t)$ is then used for a new iteration. As $k \to \infty$, the iterations are expected to converge to a fixed point, i.e. a triple (q^*, a^*, η^*) that simultaneously satisfies the HJB equation (3.18) and transport equation (3.21).

The basic strategy we employ is a finite-difference scheme which replaces derivatives with discretized increments of the respective functions over a grid. For the latter purpose we restrict to a bounded time-space domain $[0,T]\times[0,X_{max}]$ and create a gridded partition. Specifically, we partition the space domain $[0, X_{max}]$ using a mesh $0 = x_0 < x_1 < ... < x_M = X_{max}$, with equal mesh size $\Delta x = x_m - x_{m-1}, m = 1, \ldots, M$ and the time domain $[0, T]$ using a mesh $0 = t_0 < t_1 <$

... $< t_N = T$ with $t_n = n\Delta t$. The mesh sizes $\Delta x = 0.1, \Delta t = 0.01$ are fixed upfront and re-used in all the different computations below. Particularly, for transport equation (3.17a) the values $\eta(t,\cdot)$ and $q(t,\cdot)$ at $x = 0+$ are numerically approximated by $\eta(t, x_1)$ and $q(t, x_1)$, respectively.

In Section 3.3.1, we introduce the numerical method to solve the HJB equation of a representative player's game value function with price $p(t)$ exogenously given. In Section 3.3.2, we introduce the numerical method to solve the equation (3.17) of reserves distribution controlled by the optimal (q, a) obtained in the previous step. In Section 3.3.3, we show the iterative scheme to solve the coupled HJB and transport equations. Examples of stationary mean field game are given and discussed in Section 4.1. Section 4.2 introduces numerical method and examples of fluid limit model introduced in Section 4.2. For computational purpose we need to prescribe the values of the coefficients in the system of equations and the time-space domain $[0, T] \times [0, X_{max}]$ of solutions, which are summarized in Table 3.1.

$\kappa_{1,2}$	0.1	r	0.1
$\beta_{1,2}$	1	X_{max}	80
δ	1	T	50
L	5	-	-

Table 3.1: Values of coefficients used for all examples in Section 3.3.

3.3.1 Numerical scheme for the HJB equation

In this section we solve for mean field game value function $v(t, x)$ defined by (3.8) with an exogenously given price $p(t)$. Treating $p(t)$ as exogenous allows us to avoid the production control formula in (3.9) which has a mean-field dependence via $\int_0^\infty \frac{\partial}{\partial z} v(t, z)\eta(t, dz)$. Instead we use (3.14) that only depends on the player's own reserves state x, and reduces to a standard optimal stochastic control problem. For the exploration control we work with the first order condition as in (3.13). The HJB equation (3.8) with boundary condition (3.16) is similar to the single-agent problem in [40]. [40] considered a time-stationary model which reduced the HJB equation to a first order nonlinear ordinary differential equation and was solved using a Runge-Kutta scheme. The HJB equation (3.8) differs from [40] in that it has time-dependence and hence is a genuine PDE.

We employ method of lines to solve the HJB partial differential equation numerically starting with the terminal condition $v(T, x) = 0$ for all $x \in [0, X_{max}]$. Following [16], by discretizing x variable we treat the HJB equation in the time-space domain $[0, T] \times [0, X_{max}]$ as a system of ordinary differential equations in time variable t. The space derivative of $v(t, x)$ at each space grid point x_m is approximated by a backward difference quotient $\frac{\partial}{\partial x} v(t, x_m) = \frac{v(t, x_m) - v(t, x_{m-1})}{\Delta x}$. The forward difference term $\Delta_x v(t, x_m)$ is approximated by $\Delta_x v(t, x_m) = v(t, x_{m+d}) - v(t, x_m)$ with $d = \lfloor \frac{\delta}{\Delta x} \rfloor$ so that $x_m + \delta \simeq x_{m+d}$. We solve for $v(t, x_m)$ at each space grid point x_m as an ordinary differential equation in variable t, where we take $v(t, x_{m-1})$ and $v(t, x_{m+d})$ as source terms,

$$\begin{aligned}\frac{\partial}{\partial t}v(t,x_m) = rv(t,x_m) - \frac{1}{2\beta_1}\left[\left(p(t) - \kappa_1 - \frac{\partial}{\partial x}v(t,x_m)\right)^+\right]^2 \\ - \frac{1}{2\beta_2}\left[(\lambda(t)\Delta_x v(t,x_m) - \kappa_2)^+\right]^2, \quad m = 1, ..., M-d.\end{aligned} \tag{3.30}$$

For the boundary case $m = 0$, production stops and the equation becomes

$$\frac{\partial}{\partial t}v(t,x_0) = rv(t,x_0) - \frac{1}{2\beta_2}\left[(\lambda(t)\Delta_x v(t,x_0) - \kappa_2)^+\right]^2. \tag{3.31}$$

For x large enough, saturation level of reserves is reached and no exploration effort is made, thus the term $(\lambda(t)\Delta_x v(t,x) - \kappa_2)^+$ vanishes. As in [40, 41], saturation level x_{sat} of reserves is defined to be $x_{sat}(t) := \inf\{x \geq 0 : \ a(t,x) = 0\}$. For computational purpose we prescribe the equation to take the following form for $m = M-d+1, ..., M$,

$$\frac{\partial}{\partial t}v(t,x_m) = rv(t,x_m) - \frac{1}{2\beta_1}\left[\left(p(t) - \kappa_1 - \frac{\partial}{\partial x}v(t,x_m)\right)^+\right]^2. \tag{3.32}$$

Once numerical result is obtained, we need to verify that $(\lambda(t)\Delta_x v(t,x_m) - \kappa_2)^+$ is indeed zero for $m = M-d+1, ..., M$. Otherwise, the right limit X_{max} of the space domain should be extended so as to accommodate the condition $(\lambda(t)\Delta_x v(t,x_m) - \kappa_2)^+ = 0$ for $m = M-d+1, ..., M$. In all the following numerical examples, the greatest possible saturation level is about $x = 70$, thus the $X_{max} = 80$ prescribed in Table 3.1 is large enough so that the forward delay term $(\lambda(t)\Delta_x v(t,x) - \kappa_2)^+$ vanishes for $X_{max} - \delta \le x \le X_{max}$. We use Matlab fourth order ODE solver `ode45` to solve the system (3.30)-(3.32) of ordinary differential equations for $\{v(t, x_m) : m = 0, 1, ..., M\}$.

A numerical example of the HJB equation

We now apply the numerical scheme introduced above to solve the HJB equation (3.8) of a representative player's game value function with an exogenously specified price process $\{p(t), 0 \le t \le T\}$.

To prescribe $\lambda(t)$, observe that intuitively chances of a new discovery should be proportional to the remaining reserves underground. Assuming the global exploitable reserves decrease (linearly) in time due to ongoing exploration and production, we are led to consider a linear link between t and discovery rate $\lambda(t)$:

$$\lambda(t) = \left(1 - t/\bar{T}\right)^+ .$$

The time $\bar{T}$ can be viewed as global exhaustion of the commodity.

Figure 3.1 shows the computational results of optimal production rate $q(t,x)$ and exploration effort $a(t,x)$, with a constant exogenous price $p(t) = 3, \forall t \le T$. At each t, production rate $q(t,x)$ is increasing in reserves level x, while exploration effort $a(t,x)$ is decreasing in x. The monotonicity of q and a in reserves level x is due to decreasing marginal value of reserves, which is consistent with the result in [40, 41]. Both production and exploration rates decrease in t, because the discovery rate $\lambda(t)$ per unit exploration effort is decreasing, which gives decreasing motivation for exploration. As a result the production rate goes down due to dwindling reserves. The above $q(t,x)$ and $a(t,x)$ for $0 \le t \le T$ and $0 \le x \le X_{max}$ obtained in the example will be used in the next Section 3.3.2 as input to compute the evolution of reserves distribution.

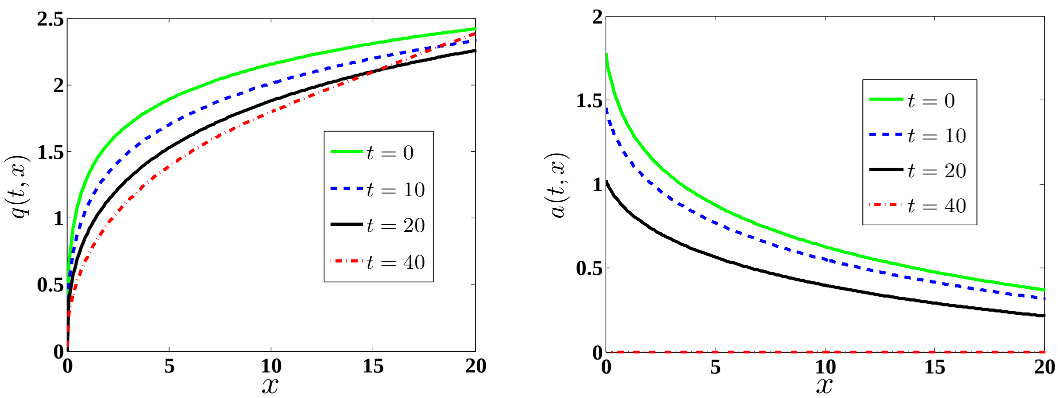

Figure 3.1: Production and exploration controls (q, a) associated with the HJB equation (3.8) under constant price $p(t) = 3$ and $\lambda(t) = (1 - 0.025t)^+, 0 \le t \le T$. Left panel: production rate $q(t,x)$. Right panel: optimized exploration rate $a(t,x)$.

3.3.2 Numerical scheme for transport equation

We now assume given controls $q(t,x), a(t,x)$ and take up the evolution of the reserves distribution. To numerically solve the transport equations of $\pi(t)$ and $\eta(t,x)$ we approximate the derivative in time by forward difference quotient and the derivative in space by forward difference quotient

$$\frac{\partial}{\partial t}\eta(t_n,x_m) \approx \frac{\eta(t_{n+1},x_m)-\eta(t_n,x_m)}{\Delta t}, \quad \frac{\partial}{\partial x}\eta(t_n,x_m) \approx \frac{\eta(t_n,x_{m+1})-\eta(t_n,x_m)}{\Delta x}.$$

By choosing $d = \lfloor \frac{\delta}{\Delta x} \rfloor$, so that $x_m - \delta \simeq x_{m-d}$ we approximate the integral term in (3.17c) with a Riemann sum

$$\int_{x_m-\delta}^{x_m} \lambda(t)a(t,x)\eta(t,dx) \approx \sum_{j=m-d+1}^{m} \lambda(t_n)a(t_n,x_j)\left(\eta(t_n,x_j)-\eta(t_n,x_{j-1})\right). \quad (3.33)$$

We start with given initial condition $\eta(t_0,x_m) = \eta_0(x_m)$, $m = 0,\ldots,M$, and $\pi(t_0) = 1-\eta(t_0,x_1)$, and solve forward in time. We also prescribe right boundary condition $\eta(t_n,x_M) = 0$, $n = 0,...,N$ with right limit X_{max} of space domain chosen large enough, which is reasonable as reserves distributions $\eta(t,x), \forall t$ in our numerical examples are compactly supported on a subset of $[0,10]$ that is included in $[0,X_{max}]$. The value of $\eta(t_n,\cdot)$ at $x=0+$ is approximated by $\eta(t_n,x_1)$, and thus the boundary probability $\pi(t_n)$ is approximated by $\eta(t_n,x_0)-\eta(t_n,x_1) \equiv 1-\eta(t_n,x_1)$. For $\eta(t_n,\cdot)$ we solve forward in space, splitting into cases according to $x_m \lesseqgtr \delta$, cf. (3.17b)-(3.17c). For $0 < x_m \leq \delta$ we obtain the numerical value of

$\eta(t_{n+1}, x_m)$ as

$$\eta(t_{n+1}, x_m) = \eta(t_n, x_m) + \Delta t q(t_n, x_m)\frac{\eta(t_n, x_{m+1}) - \eta(t_n, x_m)}{\Delta x}$$
$$- \Delta t \sum_{j=1}^{m} \lambda(t_n) a(t_n, x_{j-1}) \left(\eta(t_n, x_j) - \eta(t_n, x_{j-1})\right), \tag{3.34}$$

where we note that the first term of the finite series in (3.34) is $-\lambda(t_n)a(t_n, x_0)\pi(t_n)$, as we approximate $\pi(t_n)$ by $\pi(t_n) = \eta(t_n, x_0) - \eta(t_n, x_1)$. For $x_M > x_m > \delta$ we obtain the numerical value of $\eta(t_{n+1}, x_m)$ by

$$\eta(t_{n+1}, x_m) = \eta(t_n, x_m) - \Delta t \sum_{j=m-d+1}^{m} \lambda(t_n) a(t_n, x_{j-1}) \left(\eta(t_n, x_j) - \eta(t_n, x_{j-1})\right)$$
$$+ q(t_n, x_m) \left(\eta(t_n, x_{m+1}) - \eta(t_n, x_m)\right) \frac{\Delta t}{\Delta x}. \tag{3.35}$$

According to (3.17a), we determine the boundary probability $\pi(t_{n+1})$ by $\pi(t_{n+1}) = 1 - \eta(t_{n+1}, x_1)$.

Illustrating the Evolution of Reserves Distribution

As an example suppose that the initial reserves distribution has a parabolic initial density $m_0(x)$

$$m_0(x) = \frac{6x(u - x)}{u^3}, \quad \text{for} \quad 0 \le x \le u,$$

and $m_0(x) = 0$ otherwise. In the example shown in Figure 3.2, we take $u = 10$, cf. $m(0, x)$ on the left panel of the Figure. The evolution of boundary probability $\pi(t) = \mathbb{P}(X_t = 0)$ and the density of reserves distribution $m(t, x) = -\frac{\partial}{\partial x}\eta(t, x)$

are shown in Figure 3.2. Numerically the density function is approximated by a forward difference quotient $m(t_n, x_m) = -\frac{\eta(t_n,x_{m+1})-\eta(t_n,x_m)}{\Delta x}$.

Since discovery rate $\lambda(t)$ decreases in time, the reserves density $m(t,x)$ shifts towards zero as time evolves, as shown on the left panel of Figure 3.2. Similarly, the proportion $\pi(t)$ of producers with no remaining reserves increases in t and zero global reserves are left shortly after discovery becomes impossible $\inf\{t : \pi(t) = 0\} \simeq 41$, cf. right panel of Figure 3.2.

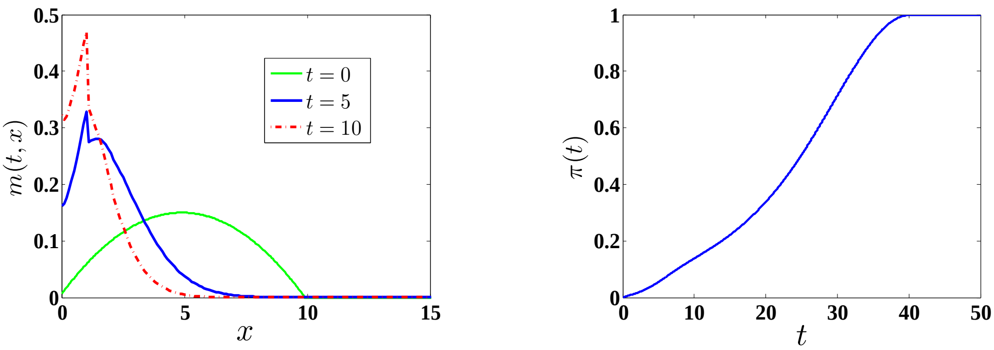

Figure 3.2: Evolution of reserves distribution under the production and exploration controls (q, a) obtained in Section 3.3.1. The discovery rate is $\lambda(t) = (1 - 0.025t)^+$ and unit amount of a discovery is $\delta = 1$. Left panel: Density of reserves distribution $m(t,x) = -\frac{\partial}{\partial x}\eta(t,x)$ for several t's. Right panel: Proportion of producers with no reserves $\pi(t) = \mathbb{P}(X_t = 0)$.

3.3.3 Numerical scheme for the MFG system

We introduce an iterative scheme to solve the system of coupled HJB and transport equations. Our solution strategy consists of a loop over the following 3 steps. To initialize, we start with a guess of price $p^{(0)}$ that is greater than κ_1 for production rate to be strictly positive.

In Step 1, given the current guess of price $p^{(k)}$, the numerical scheme in Section 3.3.1 is implemented to compute the HJB equation and obtain the optimal production $q^{(k)}$ and exploration $a^{(k)}$ rates. Next in Step 2, the production rate $q^{(k)}$ and exploration rate $a^{(k)}$ obtained from Step 1 are taken into the transport equation to compute $\eta^{(k)}$, following the numerical scheme in Section 3.3.2. Once we obtain the reserves distribution $\eta^{(k)}$, we can then compute the total production $Q^{(k)}$ by using Riemann sum to approximate the integration of $q^{(k)}(t,x)$ with respect to $\eta^{(k)}(t,\cdot)$, and then update the price to $p^{(k+1)}$ in Step 3. For each iteration k, if $p^{(k)}(t)$ is lower than equilibrium price $p(t)$ for all $t \in [0,T]$, the resulting $Q^{(k)}(t)$ will be lower than the equilibrium $Q(t)$ and the price determined by inverse demand $D^{(-1)}(Q^{(k)}(t))$ will be higher than the equilibrium price $p(t)$, and vice versa. Thus in Step 3 we take $p^{(k+1)}(t)$ in the next iteration to be the average of $p^{(k)}(t)$ and $D^{(-1)}(Q^{(k)})(t)$, so as to make sure that $p^{(k)}(t)$ converges to equilibrium price $p(t)$ as k increases. This procedure is then looped over the iterations $k = 0, 1, \ldots$ until numerical convergence.

Step 0. Start with an initial guess $p^{(0)}(t)$ of market price.

Step 1. For iteration $k = 0, 1, 2, ...$, and given $p^{(k)}(t)$, solve the HJB equation (3.18) to obtain $v^{(k)}(t, x)$ and the corresponding $q^{(k)}(t, x)$ and $a^{(k)}(t, x)$ as in (3.19)-(3.20).

Step 2. With the above $q^{(k)}$ and $a^{(k)}$ solve the transport equation to obtain $\eta^{(k)}(t, x)$ and $\pi^{(k)}(t)$ satisfying (3.17).

Step 3. Update the market price via the new total quantity of production

$$p^{(k+1)}(t) = \frac{D^{-1}\left(Q^{(k)}(t)\right) + p^{(k)}(t)}{2} \quad \text{with} \qquad Q^{(k)}(t) = -\int_0^\infty q^{(k)}(t, x)\, \eta^{(k)}(t, dx).$$

Repeat Steps 1 - 3 until convergence in the sup-norm defined as $\|\cdot\|_\infty := \sup_{[0,T]\times[0,X_{max}]} |\cdot|$. Iteration will stop when tolerance of error $TolError$ is satisfied

$$\left\|v^{(k+1)} - v^{(k)}\right\|_\infty < TolError, \quad \text{and} \quad \left\|\eta^{(k+1)} - \eta^{(k)}\right\|_\infty < TolError. \qquad (3.36)$$

In each iteration k, if $v^{(k)}(t, x)$ is lower than the equilibrium level $v(t, x)$ for all $x \in [0, X_{max}]$ with some t fixed, then in the next iteration $v^{(k+1)}(t, x)$ will move up towards the equilibrium level $v(t, x)$, and vice versa. This is illustrated in Figure 3.3 where $v^{(k)}(t, x)$ is shown starting with initial price process $p^{(0)}(t) = 3 \forall t$. As can be observed $v^{(k)}(t, x)$ converges monotonically in k pointwise at each x with $t = 10$ fixed.

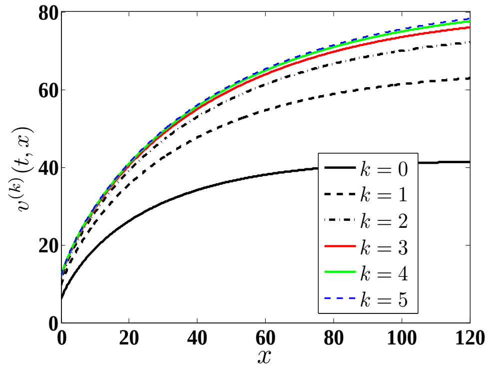

Figure 3.3: Convergence of the numerical scheme in Section 3.3.3. We start with initial guess $p^{(0)}(t) = 3 \forall t \in [0, T]$, and discovery rate $\lambda(t) = (1-0.025t)^+$. Game value function $v^{(k)}(t, x)$ converge after $k \geq 4$ iterations, with $t = 10$ fixed.

Illustration

We continue with the running example where the discovery rate is $\lambda(t) = (1 - 0.025t)^+$ and $\delta = 1$. The solutions obtained in Sections 3.3.1 and 3.3.2 can be viewed as the first iteration $k = 0$ of the above scheme. After 3 iterations, as shown in Figure 3.3, we achieve our tolerance of $TolError = 10^{-6}$ in (3.36).

To analyze the results, we compare them with the non-exploration (NE) case with zero discovery rate $\lambda(t) = 0$. When $\lambda(t) = 0$, no exploration effort will be made $a^*(t, x) \equiv 0$ as there is no hope to have any discovery. Figure 3.4 shows the resulting evolution of total production $Q(t)$, total discovery rate $A(t)$, and total reserves level $R(t)$. Total reserves $R(t)$ decrease as production proceeds; in turn decreasing $R(t)$ lowers the total production rate $Q(t)$ and raises market price $p(t)$. Interestingly we observed a hump shape in $t \mapsto A(t)$: initially exploration efforts rise, then peak and gradually decline. This complex relationship is driven by the changing exploration success parameter $\lambda(t)$ (that discourages exploration as time progresses) and the reserves distribution $\eta(t,x)$ (which encourages exploration as reserves tend to get depleted on average).

With exploration (superscript E), we confirm the intuitive relationship $R^E(t) \geq R^{NE}(t)$ which brings down the marginal value of reserves, and thus boosts production, $Q^E(t) \geq Q^{NE}(t) \forall t$. The right panel of Figure 3.5 quantifies this effect by plotting the respective $\eta(t,x)$'s. Starting with the same initial distribution at $t = 0$, the exhausted proportion $\pi^{NE}(t)$ of the case without exploration is higher than $\pi^E(t)$. In particular, global exhaustion takes hold much sooner (around $t = 10$ for $\lambda \equiv 0$ compared to $t \simeq 40$ before), cf. left panel of Figure 3.5.

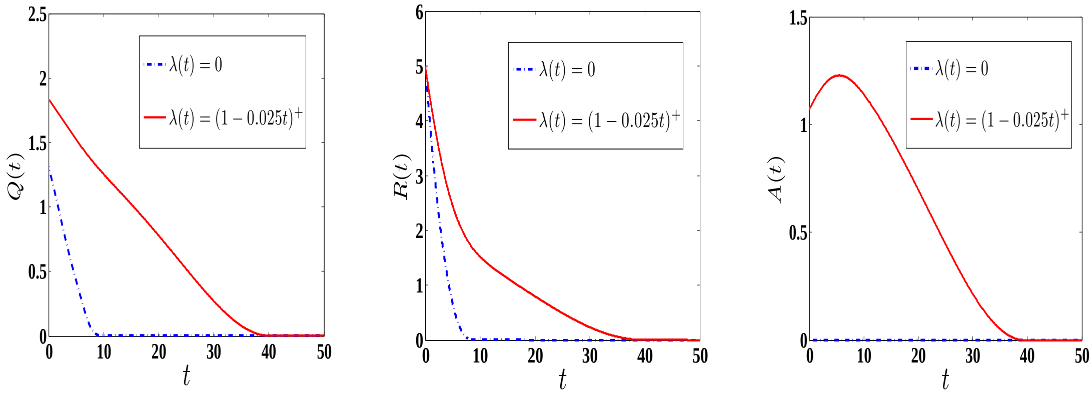

Figure 3.4: Evolution of total production $Q(t)$, total reserves $R(t)$, and total discovery rate $A(t)$ as a function of t. We compare exploration with discovery rate $\lambda(t) = (1 - 0.025t)^+$, in comparison to zero discovery rate $\lambda(t) = 0$.

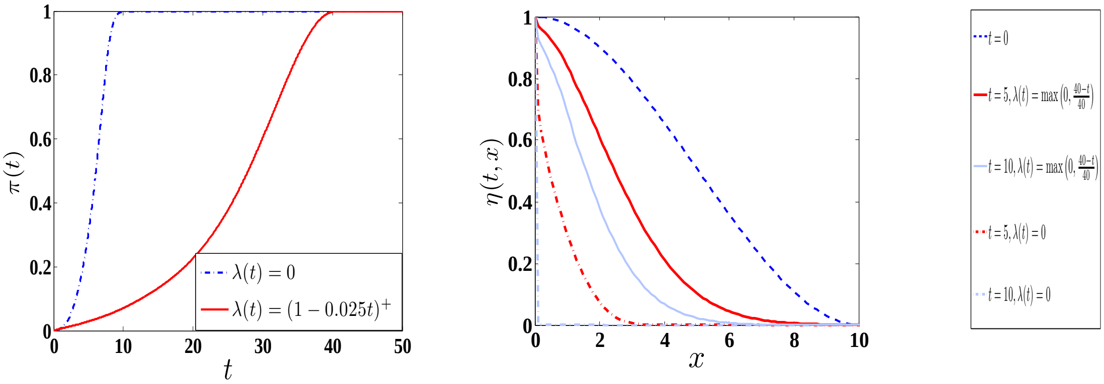

Figure 3.5: Evolution of reserves distribution with exploration $\lambda(t) = (1 - 0.025t)^+$, in comparison with no-exploration $\lambda(t) = 0$. Left Panel: Proportion $\pi(t)$ of producers with no reserves at time t. Right panel: Proportion $\eta(t, x)$ of players at time t with reserves level greater than x.

Chapter 4

Stationary Mean Field Games

In Section 3.2 we studied a generic model with time-inhomogeneous discovery rate $\lambda(t)$, which would typically be taken to be decreasing in time. When there are still abundant resources underground, it is reasonable to assume that the discovery rate is time-homogeneous $\lambda(t) = \lambda$, for some $\lambda > 0$. Thanks to exploration, the commodity used up for production can be compensated by new discoveries, and thus a *stationary* level of production and exploration can be obtained. Specifically, we expect a convergence of the reserves process X_t to a long-run stationary equilibrium, similar to the behavior of a classical uncontrolled Markov chain. In the following Section 4.1, we study the stationary MFG model, in which the reserves level remains invariant due to the counteracting effects of production and exploration. In Section 4.2, we further study the impact of uncertainty in the regime that the exploration process becomes asymptotically deterministic, so that discovery of new resources happens at high frequency with small amount of each discovery.

4.1 Stationary mean field game Nash equilibrium

In this section, we aim to search for stationary MFG equilibrium $(\tilde{q}, \tilde{a}, \tilde{\eta})$. Specifically, if the reserves has initial distribution $X_0 \sim \tilde{\eta}$, and all the players apply the strategy $q_t = \tilde{q}(x; \tilde{\eta})$ and $a_t = \tilde{a}(x; \tilde{\eta})$, then the reserves process

$$dX_t = -\tilde{q}(X_t)\mathbb{1}_{\{X_t>0\}}dt + \delta d\tilde{N}_t \tag{4.1}$$

has the distribution $\tilde{\eta}(\cdot)$ for all $t > 0$, that is, the reserves distribution is invariant in time. We define the stationary objective functional $\tilde{\mathcal{J}}$ of a player with current reserves level x and conditionally on a reserves distribution $\tilde{\eta}(\cdot)$ as

$$\tilde{\mathcal{J}}(\tilde{q}, \tilde{a}; x, \tilde{\eta}) := \mathbb{E}\left\{ \int_0^\infty \left[D^{-1}\left(\tilde{Q}(\tilde{\eta})\right) \tilde{q}(X_t) - C_q(\tilde{q}(X_t)) - C_a(\tilde{a}(X_t))\right] e^{-rt}dt \,\middle|\, X_0 = x \right\}, \tag{4.2}$$

where $\tilde{Q}(\tilde{\eta}) := -\int_0^\infty \tilde{q}(x)\tilde{\eta}(dx)$ is the stationary aggregate production.

Definition 4.1 (Stationary MFG MNE)**.** *Stationary mean field game Nash equilibrium is a triple* $(\tilde{q}^*, \tilde{a}^*, \tilde{\eta}^*)$ *such that for* (X_t) *from* (4.1) *the distribution of* *reserves* $\tilde{\eta}^* = \mathbb{P}(X_t \geq x)\forall t$ *is unchanged under the strategies* $(\tilde{q}^*, \tilde{a}^*)$*, and*

$$\tilde{v}(x) \equiv \tilde{\mathcal{J}}(\tilde{q}^*, \tilde{a}^*; \tilde{\eta}^*) \geq \tilde{\mathcal{J}}(q, a; \tilde{\eta}^*), \quad \forall (q, a) \in \mathcal{A}. \tag{4.3}$$

The following Proposition 4.1 gives the system of stationary HJB and transport equations for $\tilde{v}, \tilde{\eta}$ under a constant discovery rate $\lambda > 0$. Intuitively, it is equivalent to the equations in the previous section after dropping the dependence on t. Consequently, we pass from PDEs to ordinary differential equations in x.

Proposition 4.1 (Stationary mean field game partial differential equations)**.** *The stationary value function* $\tilde{v}$ *and upper-cumulative distribution function* $\tilde{\eta}$ *satisfy:*

$$r\tilde{v}(x) = \left[-C_a(\tilde{a}^*(x)) + \tilde{a}^*(x)\lambda\Delta_x\tilde{v}(x)\right] + \left[\tilde{p}\tilde{q}^*(x) - C_q(\tilde{q}^*(x)) - \tilde{q}^*(x)\tilde{v}'(x)\right], \quad x > 0; \tag{4.4}$$

$$\begin{cases} \tilde{\pi} = 1 - \tilde{\eta}(0+); \\ 0 = \lambda\tilde{a}^*(0)\tilde{\pi} - \int_{0+}^{x} \lambda\tilde{a}^*(z)\tilde{\eta}(dz) + \tilde{q}^*(x)\tilde{\eta}'(x), \qquad 0 < x \leq \delta, \\ 0 = -\int_{x-\delta}^{x} \lambda\tilde{a}^*(z)\tilde{\eta}(dz) + \tilde{q}^*(x)\tilde{\eta}'(x), \qquad x > \delta, \end{cases} \tag{4.5}$$

where the optimal production and exploration rates ($\tilde{q}^*$, $\tilde{a}^*$) *and the equilibrium price* $\tilde{p}$ *in stationarity are given by*

$$\begin{aligned} \tilde{q}^*(x) &= \frac{1}{\beta_1}\left(L - \tilde{Q} - \kappa_1 - \tilde{v}'(x)\right)^+, \\ \tilde{a}^*(x) &= \frac{1}{\beta_2}\left(\lambda\Delta_x\tilde{v}(x) - \kappa_2\right)^+, \\ \tilde{p} &= D^{-1}\left(\tilde{Q}\right) = L + \int_0^\infty \tilde{q}^*(x)\tilde{\eta}(dx), \end{aligned} \tag{4.6}$$

with $\tilde{Q}$ uniquely determined by the equation

$$\tilde{Q} = -\int_0^\infty \frac{1}{\beta_1}\left(L - \kappa_1 - \tilde{v}'(x) - \tilde{Q}\right)^+ \tilde{\eta}(dx).$$

Similar to [40], the boundary condition $\tilde{v}(0)$ is determined by

$$\tilde{v}(0) = \sup_{a\geq 0} \mathbb{E}\left[e^{-r\tau}\tilde{v}(\delta) - \int_0^\tau e^{-rt}C_a(a)dt\right] = \sup_{a\geq 0} \frac{a\lambda\tilde{v}(\delta) - C_a(a)}{r + a\lambda}. \tag{4.7}$$

In Section 4.1 we will introduce a numerical method to solve for stationary mean field game equilibrium.

Remark 4.1. *If the rate of new discoveries is zero, $\lambda = 0$ then from the transport equation* (4.5) *we have $\tilde{\eta}'(x) = 0$ for all $x > 0$, which implies that there is no player with positive reserves level in the long run.*

Numerical scheme and example for the stationary MFG

For the stationary MFG developed in (4.4)-(4.5) introduced, the iterative scheme introduced in section 3.3.3 is not directly applicable. The challenge lies in solving the stationary transport equation (4.5) which requires to specify the boundary condition $\tilde{\pi}$. However, $\tilde{\pi}$ is implicitly determined by the stationary transport equation itself, that is, the stationary transport equation (4.5) alone does not provide enough information about the stationary boundary condition $\tilde{\pi}$.

To overcome this issue, we utilize the non-stationary formulation (3.18)-(3.21) with a large horizon T. The basic idea is that as $T \to \infty$, the respective solution should converge to the stationary one. A related approach was taken in Chan and Sircar [17] where the stationary MFG solution was obtained by solving non-stationary transport equation coupled with stationary HJB equation and taking the large time limit. Furthermore, it is shown in [9] that a non-stationary MFG model defined on a time interval $[0, T]$ with non-local coupling converges to the corresponding stationary MFG asymptotically as $T \to \infty$. Similar convergence results are also given in [8] for the case of local coupling. According to [9], for each $t \in [0, T]$, the solution $(v(t, x), \eta(t, x))$ of a non-stationary MFG model converges in L^2-norm to the solution $(\tilde{v}(x), \tilde{\eta}(x))$ of stationary MFG model as $T \to \infty$, and the difference between stationary and non-stationary mean field game equilibrium solutions, measured by L^2-norm, is minimized at $t = T/2$.

In light of this result, we can obtain an approximate solution of the stationary MFG MNE by solving the non-stationary equations (3.18) and (3.21) with constant discovery rate $\lambda(t) \equiv \lambda$, employing the same iterative scheme as in Section 3.3.3. Then the solution $(v(t, x), \eta(t, x))$ at $t = T/2$ is taken as approximate solution of the stationary mean field game model (4.4)-(4.5), i.e., $\tilde{v}(x) \approx v(T/2, x)$ and $\tilde{\eta}(x) \approx \eta(T/2, x)$ for all $x \in [0, X_{max}]$.

A numerical example of mean field game with constant discovery rate $\lambda(t) = \lambda \equiv 1$ is shown in Figure 4.1. We take $T = 50$ and the intermediate solution $(v(t, x), \eta(t, x)) \approx (\tilde{v}(x), \tilde{\eta}(x))$ at $t = T/2 = 25$ as an approximation to the corresponding time-stationary MFG. The stationary reserves density $\tilde{m}$ is approximated as $\tilde{m}(x) \approx m(25, x) = \frac{\partial}{\partial x}\eta(25, x)$ and shown in the upper left panel of Figure 4.1. We observe that $\tilde{m}(x)$ increases in x for $0 < x \leq \delta$ where the rate of discovery is higher than the rate of production; and $\tilde{m}(x)$ decreases for $x > \delta$. The lower panels of Figure 4.1 show the evolution of total production $Q(t)$, total discovery $A(t)$, and total reserves level $R(t)$, which are defined by (3.22)-(3.24). Then the stationary total production $\tilde{Q}$, total discovery $\tilde{A}$, and total reserves level $\tilde{R}$ can be obtained approximately at $t = T/2 = 25$. The Figure confirms the link between the stationary solution and the time-dependent one. Namely we observe a boundary layer for small t (roughly $t \in [0, 12]$) arising from the non-equilibrium initial distribution $\eta_0(dx)$, and another boundary layer (roughly for $t \in [45, 50]$) arising from the terminal condition $v(T, x) = 0$. The latter causes $\lim_{t \to T} R(t) = 0, \lim_{t \to T} A(t) = 0$ observed in the plots. At the same time, for the intermediate t's all the quantities are effectively time-independent and hence should be close to the stationary MFG equilibrium solution. In particular, due to the conservation of reserves $\tilde{Q} = \tilde{A}$ we observe that $Q(t) \simeq A(t)$ in the range $t \in [15, 40]$.

It is interesting to study the effect of exploration on the equilibrium of the stationary mean field game. Figure 4.2 shows the effect of discovery rate λ on the stationary total production $\tilde{Q}$, total discovery rate $\tilde{A}$, and total reserves $\tilde{R}$, all of

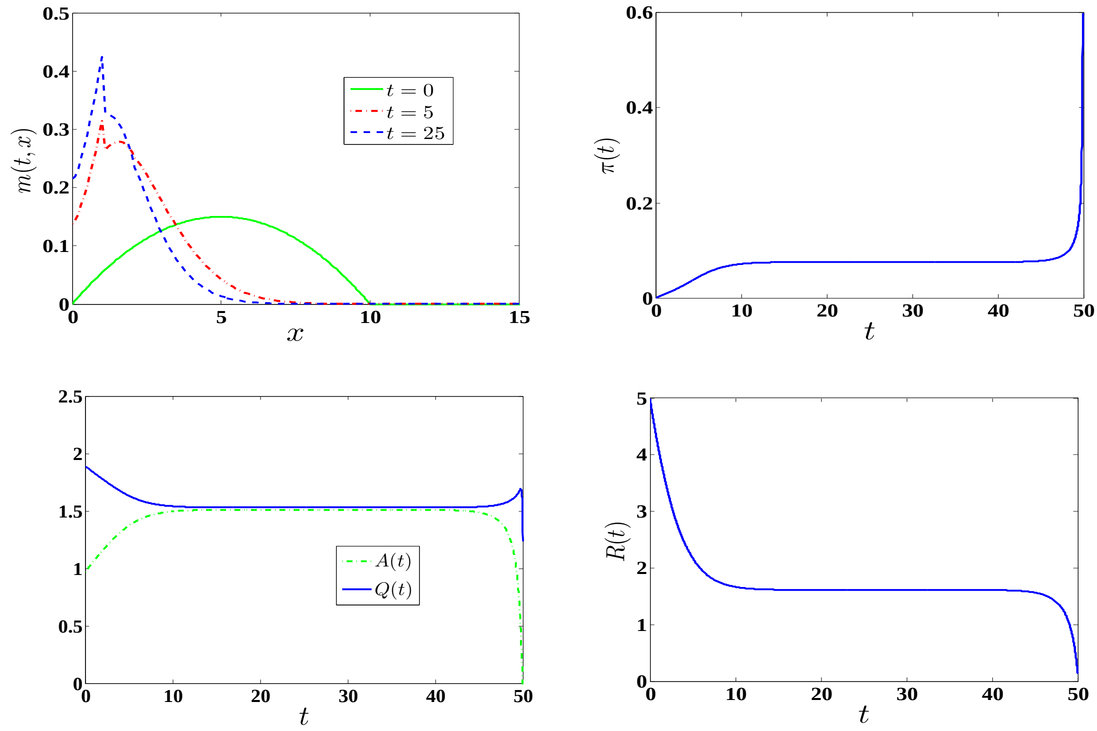

Figure 4.1: MFG solution with a constant $\lambda(t) \equiv \lambda = 1$ to illustrate the relationship between the time-dependent and stationary solutions. Upper left panel: Density $m(t, x)$ of reserves distribution. Upper right: Proportion $\pi(t)$ of producers without reserves. Lower left: Total exploration rate $A(t)$ and total production $Q(t)$. Lower right: Total reserves $R(t)$.

which have positive relation with λ. Greater λ implies greater chance to make new reserves discovery with same exploration effort, and thus higher stationary total production, cf. left panel of Figure 4.2. At low reserves level, exploration effort $\tilde{a}(x)$ increases in λ, as shown in left panel of Figure 4.3. When reserves level is large enough, as λ increases, exploration effort increases for small λ but decreases for large λ. When λ is small, e.g. $0 \le \lambda \le 0.5$ in the right panel of Figure 4.3, it is relatively hard to obtain new reserves, thus increasing λ motivates exploration effort and saturation level $\tilde{x}_{sat}$ increases in λ. As in [40, 41], the stationary saturation level $\tilde{x}_{sat}$ is defined as $\tilde{x}_{sat} := \inf\{x \ge 0 : \ \tilde{a}(x) = 0\}$. For λ large enough, e.g. $0.5 \le \lambda$ in the right panel of Figure 4.3, it becomes easy to obtain new reserves and exploration is not needed at large reserves level, thus saturation level $\tilde{x}_{sat}$ decreases in λ. As a netted effect, total discovery rate $\tilde{A}$ increases in λ. Conversely, the stationary proportion $\tilde{\pi}$ of exhausted producers decreases in λ, as expected time until next discovery at $x = 0$ shrinks due to $\lambda\tilde{a}^*(0)$ increasing, cf. right panel of Figure 4.2. When λ is very small, e.g. $\lambda < 0.05$ in Figure 4.2, exploration stops ($\tilde{A} = 0$) and reserves level becomes zero ($\tilde{R} = 0$) in stationary equilibrium. This occurs because when $\kappa_2 > 0$ and λ is small enough, the expected addition of value $\lambda\Delta_x\tilde{v}(x)$ is smaller than the cost κ_2 and thus no exploration efforts will be made. Thus, when discoveries are “too difficult”, exploration will cease even if there are still potential new reserves remaining underground, $\lambda > 0$. When

λ is small, it is hard to obtain reserves, thus increasing λ motivates exploration effort significantly. As a result, increased discovery makes reserves distribution more dispersed. The degree of dispersion of reserves distribution, measured by standard deviation $Stdev(\tilde{X}) = \sqrt{-\int_0^\infty x^2 \tilde{\eta}(dx) - \left(-\int_0^\infty x\tilde{\eta}(dx)\right)^2}$, increases in λ for small values $\lambda \in [0, 4]$, as shown in the right panel of Figure 4.4. When λ is large enough, producers do not need too many reserves as they are easy to obtain, thus probability of having a lot of reserves goes down, though total reserves level goes up. As a result, reserves distribution becomes more concentrated for discovery rate λ large enough, as shown in the right panel of Figure 4.4 that standard deviation of reserves distribution decreases in λ for $\lambda > 4$.

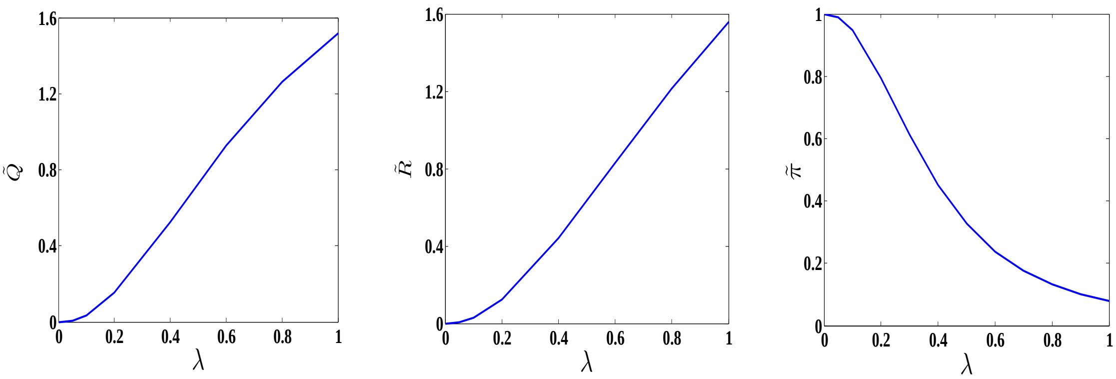

Figure 4.2: Stationary MFG solution as a function of discovery rate λ. Left panel: Total production $\tilde{Q}$. Middle: Stationary reserves $\tilde{R}$; Right: Stationary proportion $\tilde{P}$ of producers with no reserves.

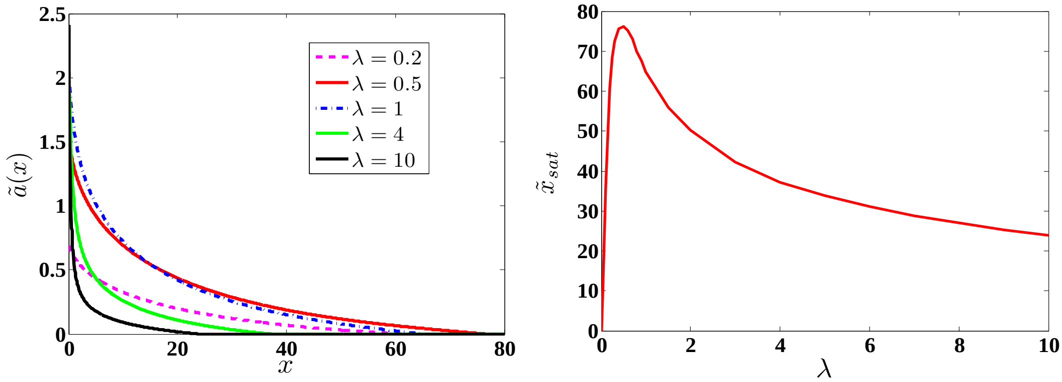

Figure 4.3: Effect of discovery rate λ on exploration effort. Left panel: Stationary exploration effort $\tilde{a}(x)$ with different values of λ. Right: Stationary saturation level $\tilde{x}_{sat} = \inf\{x \geq 0 : \ \tilde{a}(x) = 0\}$ as a function of λ.

4.2 Fluid limit of exploration process

In this section we study the effect of randomness of exploration process on equilibrium production and reserves distribution. The stochasticity of the exploration process depends on two factors: the discovery rate λ per unit exploration effort, and the size δ of each discovery. It is interesting to study the asymptotic behavior of producers as the game becomes more deterministic. To do so, we introduce an asymptotic parameter $\epsilon > 0$, rescaling $\lambda_\epsilon := \lambda/\epsilon$ and $\delta_\epsilon := \delta\epsilon$. As $\epsilon \downarrow 0$, we have the discovery rate $\lambda_\epsilon \uparrow \infty$ and unit discovery amount $\delta_\epsilon \downarrow 0$, which means that the exploration process becomes more deterministic.

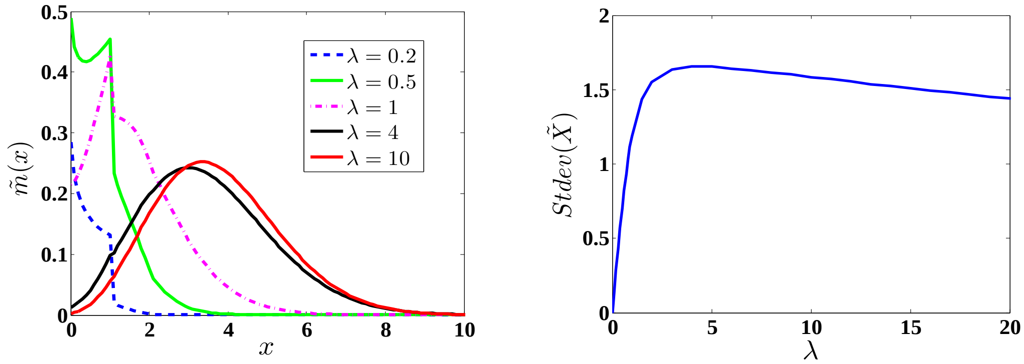

Figure 4.4: Effect of discovery rate λ on reserves distribution. Left panel: Density $\tilde{m}(x)$ of stationary reserves distribution with different values of λ. Right: Standard deviation $Stdev(\tilde{X})$ of stationary reserves distribution against discovery rate λ, where $Stdev(\tilde{X}) = \sqrt{-\int_0^\infty x^2 \tilde{\eta}(dx) - \left(-\int_0^\infty x\tilde{\eta}(dx)\right)^2}$.

We aim to study the asymptotic behavior, as $\epsilon \downarrow 0$, of the equilibrium production and exploration $(\tilde{q}_\epsilon, \tilde{a}_\epsilon)$ and reserves distribution $\tilde{\eta}_\epsilon$ associated with the game value function $\tilde{v}_\epsilon$. For the limiting case $\epsilon = 0$ when the exploration process is fully deterministic, the non-stationary mean field game equations are given by (4.8)-(4.9). Intuitively, the difference term $\Delta_x v(t,x) = v(t, x+\delta) - v(t,x)$ becomes $\frac{\partial}{\partial x} v_0(t,x)$ in the fluid limit and the integral becomes $\delta a_0^*(t,x) \frac{\partial}{\partial x} \eta_0(t,x)$.

$$0=\frac{\partial}{\partial t}v_0(t,x)-rv_0(t,x)+\frac{1}{2\beta_1}\left[\left(p(t)-\kappa_1-\frac{\partial}{\partial x}v_0(t,x)\right)^+\right]^2$$
$$+\frac{1}{2\beta_2}\left[\left(\lambda\delta\frac{\partial}{\partial x}v_0(t,x)-\kappa_2\right)^+\right]^2. \tag{4.8}$$

$$\begin{cases}\pi_0(t)=1-\eta_0(t,0+);\\ \frac{\partial}{\partial t}\eta_0(t,x)=\left(-\lambda\delta a_0^*(t,x)+q_0^*(t,x)\right)\frac{\partial}{\partial x}\eta_0(t,x), \quad x>0,\end{cases} \tag{4.9}$$

where the optimal production rate q_0^* and exploration rate a_0^* are

$$q_0^*(t,x)=\arg\max_{q\geq 0}\left[p_0(t)q(t,x)-C_q(q(t,x))-q(t,x)\frac{\partial}{\partial x}v_0(t,x)\right]$$
$$=\frac{1}{\beta_1}\left(p_0(t)-\kappa_1-\frac{\partial}{\partial x}v_0(t,x)\right)^+, \tag{4.10}$$
$$a_0^*(t,x)=\arg\max_{a\geq 0}\left[-C_a(a(t,x))+a(t,x)\lambda\delta\frac{\partial}{\partial x}v_0(t,x)\right]$$
$$=\frac{1}{\beta_2}\left(\lambda\delta\frac{\partial}{\partial x}v_0(t,x)-\kappa_2\right)^+. \tag{4.11}$$

The boundary conditions $v_0(t,0)$ and $\frac{\partial}{\partial x}v_0(t,0)$ are given explicitly by the following lemma 4.1.

Lemma 4.1. *The boundary conditions $v_0(t,0)$ and $\frac{\partial}{\partial x}v_0(t,x)$ satisfy*

$$\frac{\partial}{\partial x}v_0(t,0)=\frac{\beta_2(p_0(t)-\kappa_1)+\beta_1\lambda\delta\kappa_2}{\beta_1\lambda^2\delta^2+\beta_2}; \tag{4.12}$$
$$v_0(t,0)=\int_t^T\left[\frac{\lambda\delta(p_0(s)-\kappa_1)-\kappa_2}{\beta_1\lambda^2\delta^2+\beta_2}\right]^2(1+\lambda^2\delta^2)\mathrm{e}^{-r(s-t)}ds. \tag{4.13}$$

Proof. See Appendix A.5. □

The following Proposition 4.2 summarizes the stationary MFG in the fluid limit $\epsilon = 0$.

Proposition 4.2 (Stationary mean field game equilibrium in fluid limit)**.** *The stationary mean field equilibrium in fluid limit($\epsilon = 0$) is summarized as*

(i). The stationary reserves distribution is $\tilde{\pi}_0 = 1$, i.e. all producers choose to hold no reserves, $\tilde{R}_0 = 0$.

(ii). The equilibrium total production $\tilde{Q}_0$ and market price in the fluid limit are given by

$$\tilde{Q}_0 = \frac{[(L-\kappa_1)\lambda\delta - \kappa_2]^+}{\beta_2 + (1+\beta_1)\lambda\delta}, \qquad \tilde{p}_0 = L - \tilde{Q}_0 \tag{4.14}$$

(iii). The equilibrium exploration control is $\tilde{a}_0^*(x) = 0 \; \forall x > 0$ and

$$\tilde{a}_0^*(0) = \frac{1}{\delta\lambda}\tilde{q}_0^*(0). \tag{4.15}$$

The proof of Proposition 4.2 is Appendix A.4 . In the case of fluid limit $\epsilon = 0$, discovery of new resources happens in a completely deterministic way, thus it is not necessary to hold reserves for production. Producers starting with positive reserves will not explore until reserves run out. Once reserves level reaches zero, equation (4.15) implies that a player without reserves will choose production and exploration strategies such that the production rate exactly equals the rate of reserves increment due to his exploration effort. This explains how zero reserves can be sustained in equilibrium. Overall, the above Proposition shows that the stationary equilibrium with deterministic exploration is trivial, i.e. only $x = 0$ matters and the system of ODEs effectively collapses to algebraic equations linking $\tilde{Q}_0$ and $\tilde{A}_0$ to model parameters. This shows that the stochastic model is strictly more complex than the deterministic one.

Numerical example of a fluid limit model

In Section 4.2 we studied the model where exploration was deterministic. In this section we present the effect of the introduced parameter ϵ in the regime $\epsilon \downarrow 0$.

The iterative scheme in Section 3.3.3 is easily adapted to solve the fluid limit system (4.8)-(4.9). As in Section 3.3.1, we employ method of lines to numerically solve the HJB equation. The space derivative of $v_0(t, x)$ at each space grid point x_m is approximated by a forward difference quotient $\frac{\partial}{\partial x} v_0(t, x_m) = \frac{v_0(t,x_m) - v_0(t,x_{m-1})}{\Delta x}$. At each space grid point x_m, we represent $\frac{\partial}{\partial t} v_0(t, x_m)$ in terms of $v_0(t, x_m)$ and

$\frac{\partial}{\partial x}v_0(t,x_m)$ with $v_0(t,x_{m-1})$ taken as source term,

$$\frac{\partial}{\partial t}v_0(t,x_m) = rv_0(t,x_m) - \frac{1}{2\beta_1}\left[\left(p(t)-\kappa_1-\frac{\partial}{\partial x}v_0(t,x_m)\right)^+\right]^2 - \frac{1}{2\beta_2}\left[\left(\lambda\delta\frac{\partial}{\partial x}v_0(t,x_m)-\kappa_2\right)^+\right]^2, \quad m=1,2,...,M. \tag{4.16}$$

We use Matlab fourth-order ODE solver `ode45` to solve the system (4.16) of ordinary differential equations for $\{v(t,x_m): m=0,1,...,M\}$ with boundary condition $v_0(t,x_0)\equiv v_0(t,0)$ given by (4.13) and terminal condition $v(t,x_m)=0$ for all $m=0,1,...,M$.

We use forward in time and forward in space scheme to solve the transport equation (4.9). As in section 3.3.2, we also prescribe right boundary condition $\eta_0(t_n,x_M)=0$, $n=0,...,N$ with right limit X_{max} of space domain chosen large enough, which is reasonable as reserves distribution $\eta_0(t,x),\forall t$ in the numerical example is compactly supported on a subset of $[0,10]$ that is included in $[0,X_{max}]$. Similar to (3.34), we obtain the numerical value of $\eta_0(t_{n+1},x_m)$ as

$$\eta_0(t_{n+1},x_m) = \eta_0(t_n,x_m) + \Delta t\left[-\lambda\delta a_0(t_n,x_m)+q_0(t_n,x_m)\right]\frac{\eta_0(t_n,x_{m+1})-\eta_0(t_n,x_m)}{\Delta x}, \quad m=2,...,M-1.$$

According to (4.9), we determine the boundary probability $\pi(t_{n+1})$ by $\pi(t_{n+1}) = 1 - \eta(t_{n+1}, x_1)$, where $\eta(t_{n+1}, x_1)$ is used to approximate the value of $\eta(t_{n+1}, \cdot)$ at $x = 0+$.

We find that a more deterministic discovery of resources lowers the game value as well as the marginal value of reserves. As discovery becomes more deterministic, it is less necessary to hold reserves for production. Thus the stationary reserves level $\tilde{R}_\epsilon$ decreases as $\epsilon \downarrow 0$, as shown in the right panel of Figure 4.5. Deterministic discovery of resources also boosts total production as shown in left and middle panels of Figure 4.5, and consequently lowers the price.

Economically, the uncertainty in reserves replenishment forces producers to hold a buffer of current reserves. This buffer can be reduced as $\epsilon \downarrow 0$. In the limit $\epsilon = 0$, production can be viewed as a perfect just-in-time supply chain: effort is expended to find an infinitesimal amount of new underground resources which are immediately extracted and sold for profit. Thus, exploration effort becomes equivalent to a secondary production cost, the cost of securing the commodity supply to exactly match the desired production rate. Moreover, the uncertainty discourages exploration (due to the time-value of money of putting in effort today for a delayed reward at discovery date τ).

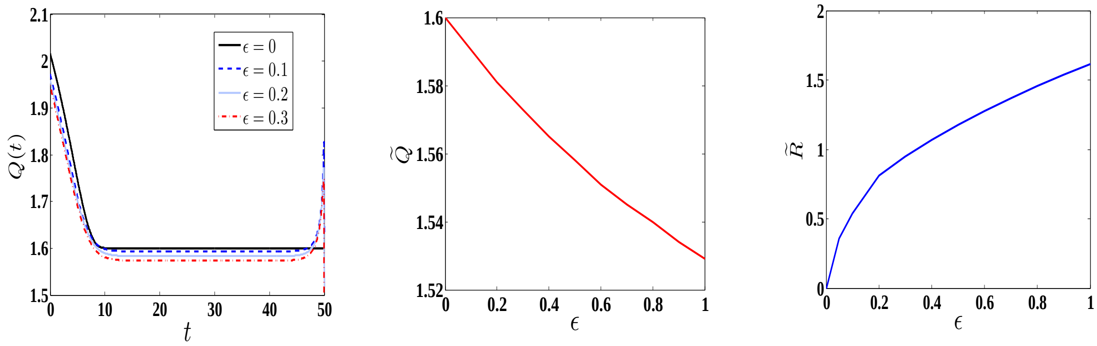

Figure 4.5: Equilibrium production and reserves level in the regime $\lambda_\epsilon = \lambda/\epsilon$ and $\delta_\epsilon = \delta\epsilon$ for different values of ϵ. Left panel: Evolution of total production $Q_\epsilon(t)$. Middle: Stationary production $\tilde{Q}_\epsilon$ against ϵ. Right : Stationary reserves level $\tilde{R}_\epsilon$ against ϵ.

Chapter 5

Conclusion and future work

5.1 Conclusion

In Chapter 2 we studied the effect of exploration and stochastic demand in dynamic Cournot games. In the model of [40], players competed in a dynamic noncooperative game as their reserves of an exhaustible resource depleted, simultaneously exploring new reserves. The only stochastic aspect was (Poissonian) randomness in reserve discoveries, making the overall game to be piecewise deterministic. The stochastic demand in our research adds a further feature of fluctuating market prices, introducing further dynamic aspects into the duopoly. We modelled this feature through a regime-switching price (inverse demand) function, which represented a random environment under which the producers make strategies of production and exploration.

Stochastic demand creates the possibility of a new (dubbed Type M2) equilibrium whereby the exhaustible producer may opportunistically shutdown production in hopes of higher profits in the future. This happens when reserves are low and their shadow marginal cost is high enough. Additionally, the non-monotonic impact of competition on exploration efforts already observed in [40] continues to occur in our model and leads to interesting phenomena herein.

In Chapters 3 and 4 we employ mean field game approach to study the production and exploration of exhaustible commodities with a large population of producers. Among versions of the model we study are: stationary and non-stationary exploration rates; stochastic and deterministic reserves discovery process. Several numerical schemes based on finite-difference approaches have been developed to analyze and illustrate the above models.

It is found that total production and total reserves level increase as discovery rate increases, as higher discovery rate boosts exploration effort and thus increases reserves for production. It is also found that when discovery rate is small enough (not necessarily zero), exploration will stop, because the expected value addition cannot cover the varying cost of exploration activity.

We also study the effect of uncertainty in the process of exploration and discovery. Uncertainty forces producers to hold a buffer of current reserves, and the buffer reduces as uncertainty reduces. It is also found that exploration is discouraged by uncertainty, due to the time-value of investment of putting in effort today for a delayed reward at a later discovery date. Deterministic exploration and discovery of resources boost total production and consequently lower the price.

5.2 Future work

A variant of the MFG approach presented in Chapters 3 and 4 would be to consider competition between a single major energy producer and a large population of minor energy producers. This would correspond for example to the dominant role played by the Organization of Petroleum Exporting Countries (OPEC) in the crude oil market, with OPEC controlling about 40% of the world's oil production. Due to the resulting market power, the minor producers choose production strategies based on the production strategy of OPEC. The corresponding game model would involve a game value function for the major player, a game value function for a representative minor producer, and the reserves distribution of minor producers. The price is then determined by the aggregate production of the major plus all minor producers.

Since the major producer has dominating power to determine price, the minor producers have to make production decision based on the major producer's strategy. Taking into account the full information about the minor producers' production strategy, the major producer decides his own production quantity in order to maximize his profit. And then the minor producers choose quantity of production based on the major producer's strategy. The major producer and minor producers realize profit under the equilibrium price determined through the above decision process.

The mean filed game model with a major player and a group of minor players consists of three equations: an HJB equation of the game value function of the major player; an HJB equation of the game value function of a representative minor player; an equation for the reserves distribution of minor players. The price, as the mean field term, is a function of the total production that is equal to the production of the major producer and the total production of all minor producers.

To numerically solve the system of equations, we can adopt the iterative scheme in Section 3.3.3. Starting with an initial guess of price, we solve the HJB equation of a minor representative producer to obtain optimal production strategy of minor producers. Then we solve the transport equation, and compute the total production of all minor producers. With the total production of all minor producers obtained, we solve the HJB equation of the major producer and obtain the optimal production strategy of the major producer. Then we update the price according to the optimal production strategies of the major producer and minor producers, and use the updated price for the next iteration until numerical convergence.

Appendix A

Appendix

A.1 Proof of Lemma 2.1

Proof. To derive the asymptotic game functions as $\lambda_{01} \to +\infty$, we set $\lambda_{01} = \frac{1}{\epsilon}$. Without loss of generality, we assume that the asymptotic expansion of $v_M(x)$ with respect to ϵ is

$$v_i^{\epsilon} = v_i^0 + g(\epsilon)v_i^1 + o(g(\epsilon)), \qquad i = L, H \tag{A.1}$$

where $g(\epsilon) \to 0$, as $\epsilon \to 0$.

We substitute the v_L and v_H in the HJB ODEs with their asymptotic expansion to obtain

$$\left[\frac{2}{3}\left(\frac{L+c}{2}-(v_L^\epsilon)'(x)\right)^+-\frac{1}{6}\left(2c-L-(v_L^\epsilon)'(x)\right)^+\right]^2+\frac{1}{\gamma}[(\lambda\Delta v_L^\epsilon(x)-\kappa)^+]^\gamma$$
$$+\frac{1}{\epsilon}\left(v_H^0(x)-v_L^0(x)\right)+\frac{1}{\epsilon}\left(g(\epsilon)[v_H^1(x)-v_L^1(x)]+o(g(\epsilon))\right)-rv_L^\epsilon(x)=0, \tag{A.2}$$

$$\left[\frac{2}{3}\left(\frac{H+c}{2}-(v_H^\epsilon)'(x)\right)^+-\frac{1}{6}\left(2c-H-(v_H^\epsilon)'(x)\right)^+\right]^2+\frac{1}{\gamma}[(\lambda\Delta v_H^\epsilon(x)-\kappa)^+]^\gamma$$
$$+\lambda_{10}\left(v_L^0(x)-v_H^0(x)\right)+g(\epsilon)\lambda_{10}(v_L^1(x)-v_H^1(x))-rv_H^\epsilon(x)+o(g(\epsilon))=0. \tag{A.3}$$

We must have that $\lim_{\epsilon\to 0} v_L^\epsilon = \lim_{\epsilon\to 0} v_L^\epsilon$, i.e. $v_L^0 = v_H^0 =: \tilde{v}$, otherwise the term $\epsilon^{-1}(v_H^0 - v_L^0)$ will explode as $\epsilon \to 0$. Making that simplification, multiplying (A.2) by $\epsilon\lambda_{10}$ and adding (A.3) we obtain

$$0=\epsilon\lambda_{10}\left[\frac{2}{3}\left(\frac{L+c}{2}-(v_L^\epsilon)'(x)\right)^+-\frac{1}{6}\left(2c-L-(v_L^\epsilon)'(x)\right)^+\right]^2$$
$$+\left[\frac{2}{3}\left(\frac{H+c}{2}-(v_H^\epsilon)'(x)\right)^+-\frac{1}{6}\left(2c-H-(v_H^\epsilon)'(x)\right)^+\right]^2$$
$$+\frac{\epsilon\lambda_{10}}{\gamma}[(\lambda\Delta v_L^\epsilon(x)-\kappa)^+]^\gamma+\frac{1}{\gamma}[(\lambda\Delta v_H^\epsilon(x)-\kappa)^+]^\gamma-r\left(\lambda_{10}\epsilon v_L^\epsilon(x)+v_H^\epsilon(x)\right). \tag{A.4}$$

One can now take $\epsilon \to 0$ which reduces to a regular perturbation of the following ODE for $\tilde{v}(x)$ (note that the first term involving L vanishes):

$$\left[\frac{2}{3}\left(\frac{H+c}{2} - (\tilde{v}_H)'(x)\right)^+ - \frac{1}{6}\left(2c - H - (\tilde{v}_H)'(x)\right)^+\right]^2$$
$$+ \frac{1}{\gamma}[(\lambda\Delta\tilde{v}_H(x) - \kappa)^+]^\gamma - r\tilde{v}_H(x) = 0,$$

which matches the solution of an exploration duopoly game studied in [40] with linear inverse demand $p_t = H - q_t^1 - q_t^2$.

For the boundary conditions, we re-write (2.5) as

$$(v_H^\epsilon(0) - v_L^\epsilon(0))\left(\frac{1}{\epsilon}\right) + v_L^\epsilon(\delta)\lambda a_L^\epsilon(0) - C_a(a_L^\epsilon(0)) - (r + \lambda a_L^\epsilon(0))\, v_L^\epsilon(0) = 0, \tag{A.5}$$

$$(v_L^\epsilon(0) - v_H^\epsilon(0))\,\lambda_{10} + v_H^\epsilon(\delta)\lambda a_H^\epsilon(0) - C_a(a_H^\epsilon(0)) - (r + \lambda a_H^\epsilon(0))\, v_H^\epsilon(0) = 0. \tag{A.6}$$

We multiply (A.5) by $\epsilon\lambda_{10}$ and add to (A.6) to obtain

$$\begin{aligned} &\epsilon\lambda_{10}\left[v_L^\epsilon(\delta)\lambda a_L^\epsilon(0) - C_a(a_L^\epsilon(0)) - (r + \lambda a_L^\epsilon(0))\, v_L^\epsilon(0)\right] \\ &+ \left[v_H^\epsilon(\delta)\lambda a_H^\epsilon(0) - C_a(a_H^\epsilon(0)) - (r + \lambda a_H^\epsilon(0))\, v_H^\epsilon(0)\right] = 0. \end{aligned} \tag{A.7}$$

Letting $\epsilon \to 0$ removes the first terms and we are left with

$$\tilde{v}(\delta)\lambda\tilde{a}(0) - C_a(\tilde{a}(0)) - (r + \lambda\tilde{a}(0))\,\tilde{v}(0) = 0,$$

which is equivalent to

$$\tilde{v}(0) = \frac{\tilde{v}(\delta)\lambda\tilde{a}(0) - C_a(\tilde{a}(0))}{(r + \lambda\tilde{a}(0))} = \sup_a \frac{\tilde{v}(\delta)\lambda a - C_a(a)}{(r + \lambda a)}, \tag{A.8}$$

again matching the corresponding boundary condition in the deterministic demand setting.

$\square$

A.2 Proof of Lemma 2.2

Proof. We set $\lambda_{01} = \frac{b_L}{\epsilon}, \lambda_{10} = \frac{b_H}{\epsilon}$, where b_L, b_H are some constants, and $\epsilon > 0$ can be arbitrarily small. The stationary distribution $\vec{\pi}$ given by $\pi_L = \frac{\lambda_{10}}{\lambda_{01}+\lambda_{10}} = \frac{b_H}{b_L+b_H}$, $\pi_H = \frac{b_L}{b_L+b_H}$ is unchanged as $\epsilon \to 0$.

We consider the asymptotic expansions of v_M in terms of ϵ:

$$v_i^\epsilon = v_i^0 + f(\epsilon)v_i^1 + o(f(\epsilon)), \qquad i = L, H, \tag{A.9}$$

where $f(\epsilon) \to 0$, as $\epsilon \to 0$. Substituting (A.9) into (2.13) yields

$$\left[\frac{2}{3}\left(\frac{L+c}{2} - (v_L^\epsilon)'(x)\right)^+ - \frac{1}{6}\left(2c - L - (v_L^\epsilon)'(x)\right)^+\right]^2 + \frac{1}{\gamma}[(\lambda\Delta v_L^\epsilon(x) - \kappa)^+]^\gamma \tag{A.10}$$

$$+\frac{b_L}{\epsilon}\left(v_H^0(x) + f(\epsilon)v_H^1(x) - v_L^0(x) - f(\epsilon)v_L^1(x)\right) - rv_L^\epsilon(x) + o(f(\epsilon)) = 0;$$

$$\left[\frac{2}{3}\left(\frac{H+c}{2} - (v_H^\epsilon)'(x)\right)^+ - \frac{1}{6}\left(2c - H - (v_H^\epsilon)'(x)\right)^+\right]^2 + \frac{1}{\gamma}[(\lambda\Delta v_H^\epsilon(x) - \kappa)^+]^\gamma \tag{A.11}$$

$$+\frac{b_H}{\epsilon}\left(v_L^0(x) + f(\epsilon)v_L^1(x) - v_H^0(x) - f(\epsilon)v_H^1(x)\right) - rv_H^\epsilon(x) + o(f(\epsilon)) = 0.$$

We must have that $\lim_{\epsilon\to 0} v_L^\epsilon = \lim_{\epsilon\to 0} v_L^\epsilon$, i.e. $v_L^0 = v_H^0 = \bar{v}$, otherwise the terms $\frac{b_L}{\epsilon}(v_H^\epsilon - v_L^\epsilon)$ and $\frac{b_H}{\epsilon}(v_L^\epsilon - v_H^\epsilon)$ above would explode as $\epsilon \to 0$. Indeed, it is clear that $|v_L(x) - v_H(x)| \to 0$ as $\epsilon \to 0$ due to the fast switching of the regimes, making the initial macroeconomic conditions irrelevant.

Canceling the terms $v_L^0 - v_H^0 \equiv 0$ in (A.10)-(A.11), multiplying (A.10) by $b_H/(b_L + b_H)$, (A.11) by $b_L/(b_L + b_H)$, and adding them up we obtain

$$
\begin{aligned}
0 = \pi_L &\left[\frac{2}{3}\left(\frac{L+c}{2} - (v_L^\epsilon)'(x)\right)^+ - \frac{1}{6}\left(2c - L - (v_L^\epsilon)'(x)\right)^+\right]^2 \\
&+ \pi_H \left[\frac{2}{3}\left(\frac{H+c}{2} - (v_H^\epsilon)'(x)\right)^+ - \frac{1}{6}\left(2c - H - (v_H^\epsilon)'(x)\right)^+\right]^2 \\
+\frac{\pi_L}{\gamma}&[(\lambda\Delta v_L^\epsilon(x) - \kappa)^+]^\gamma + \frac{\pi_H}{\gamma}[(\lambda\Delta v_H^\epsilon(x) - \kappa)^+]^\gamma - r\left(\pi_L v_L^\epsilon(x) + \pi_H v_H^\epsilon(x)\right) + o(f(\epsilon)).
\end{aligned}
$$

Note that all the terms involving ϵ^{-1} have cancelled out. Once again plugging in (A.9) we can now take $\epsilon \to 0$ since this just amounts to a regular perturbation; the result is precisely (2.19).

For the boundary conditions, we re-write the original

$$
v_i^\epsilon(0) = \frac{v_j^\epsilon(0)(\frac{b_i}{\epsilon}) + v_i^\epsilon(\delta)\lambda a_i^\epsilon(0) - C_a(a_i^\epsilon(0))}{r + \frac{b_i}{\epsilon} + \lambda a_i^\epsilon(0)}, \quad i, j = L, H
$$

as

$$\left(v_H^\epsilon(0) - v_L^\epsilon(0)\right)\left(\frac{b_L}{\epsilon}\right) + v_L^\epsilon(\delta)\lambda a_L^\epsilon(0) - C_a(a_L^\epsilon(0)) - \left(r + \lambda a_L^\epsilon(0)\right) v_L^\epsilon(0) = 0, \tag{A.12}$$

$$\left(v_L^\epsilon(0) - v_H^\epsilon(0)\right)\left(\frac{b_H}{\epsilon}\right) + v_H^\epsilon(\delta)\lambda a_H^\epsilon(0) - C_a(a_H^\epsilon(0)) - \left(r + \lambda a_H^\epsilon(0)\right) v_H^\epsilon(0) = 0. \tag{A.13}$$

Once again multiplying (A.12) by $b_H/(b_L + b_H)$ and (A.13) by $b_L/(b_L + b_H)$, and summing produces

$$\begin{aligned}\pi_L v_L^\epsilon(\delta)\lambda a_L^\epsilon(0) + \pi_H v_H^\epsilon(\delta)\lambda a_H^\epsilon(0) - \pi_L C_a(a_L^\epsilon(0)) - \pi_H C_a(a_H^\epsilon(0))&\\ - \pi_L\left(r + \lambda a_L^\epsilon(0)\right) v_L^\epsilon(0) - \pi_H\left(r + \lambda a_H^\epsilon(0)\right) v_H^\epsilon(0) = 0.&\end{aligned} \tag{A.14}$$

As $\epsilon \to 0$, $a_M^\epsilon(0) = [(\lambda\Delta v_M^\epsilon(0) - \kappa)^+]^{\gamma-1} \to [(\lambda\Delta\bar{v}(0) - \kappa)^+]^{\gamma-1} = \bar{a}(0)$, and we find $\bar{v}(\delta)\lambda\bar{a}(0) - C_a(\bar{a}(0)) - \left(r + \lambda\bar{a}(0)\right)\bar{v}(0) = 0$, which is equivalent to (2.20). □

A.3 Proof of Proposition 3.1

Proof. In any time interval $(0, T)$, let $h(t, x) \in C_c^\infty((0, T) \times \mathbb{R}_+)$ be a test function that is supported in $(0, T) \times \mathbb{R}_+$. Using Itô formula for jump processes, we have

$$\begin{aligned}0 &= h(T, X_T) - h(0, X_0)\\ &= \int_0^T \frac{\partial}{\partial t}h(t, X_t) - q(t, X_t)\frac{\partial}{\partial x}h(s, X_t)dt + \int_0^T h(t, X_t) - h(t-, X_{t-})dN_t,\end{aligned} \tag{A.15}$$

where the first equality is due to the fact that $h(\cdot,\cdot)$ has support in $(0,T)\times\mathbb{R}_+$ so that $h(T,X_T)=h(0,X_0)=0$. Apply expectation operator $\mathbb{E}[\cdot]$ to equation (A.15)

$$\begin{aligned}
0 &= \mathbb{E}\left[h(T,X_T)-h(0,X_0)\right] \\
&= \mathbb{E}\left[\int_0^T \frac{\partial}{\partial t}h(t,X_t)-q(t,X_t)\frac{\partial}{\partial y}h(t,X_t)dt+\int_0^T h(t,X_t)-h(t-,X_{t-})dN_t\right] \\
&= -\int_0^\infty\int_0^T \frac{\partial}{\partial t}h(t,x)\frac{\partial}{\partial x}\eta(t,x)dtdx+\int_0^\infty\int_0^T q(t,x)\frac{\partial}{\partial x}h(t,x)\frac{\partial}{\partial x}\eta(t,x)dtdx \\
&\quad -\int_\delta^\infty\int_0^T \left(h(t,x)-h(t,x-\delta)\right)\lambda(t)a(t,x-\delta)\frac{\partial}{\partial x}\eta(t,x-\delta)dtdx \\
&\quad +\int_0^T h(t,\delta)\lambda(t)a(t,0)\pi(t)dt =: I1+I2+I3+I4.
\end{aligned} \tag{A.16}$$

By integration-by-parts and the fact that $h(t,x)$ has compact support in $(0,T)\times\mathbb{R}_+$, the first term on the right hand side of the last equality of (A.16) equals to

$$I1=\int_0^\infty\int_0^T \eta(t,x)\frac{\partial}{\partial t}\frac{\partial}{\partial x}h(t,x)dtdx=-\int_0^\infty\int_0^T \frac{\partial}{\partial x}h(t,x)\frac{\partial}{\partial t}\eta(t,x)dtdx. \tag{A.17}$$

By defining $F(t,x) := \int_0^x \lambda(t)a(t,z)\eta(t,dz)$, $x>0$, the third term on the right hand side of the last equality of (A.16) can be written as

$$
\begin{aligned}
I3 &= -\int_\delta^\infty \int_0^T h(t,x)\frac{\partial}{\partial x}F(t,x-\delta)dtdx + \int_0^\infty \int_0^T h(t,x)\frac{\partial}{\partial x}F(t,x)dt\,dx \\
&= \int_\delta^\infty \int_0^T F(t,x-\delta)\frac{\partial}{\partial x}h(t,x)dtdx - \int_0^\infty \int_0^T F(t,x)\frac{\partial}{\partial x}h(t,x)dt \\
&= \int_\delta^\infty \int_0^T \left(F(t,x)-F(t,x-\delta)\right)\frac{\partial}{\partial x}h(t,x)dtdx - \int_0^\delta \int_0^T F(t,x)\frac{\partial}{\partial x}h(t,x)dt\,dx.
\end{aligned}
\tag{A.18}
$$

The fourth term on the right hand side of the last equality of (A.16) can be written as

$$
\begin{aligned}
I4 &= \int_0^T \left(\int_0^\delta \frac{\partial}{\partial x}h(t,x)dx\right)\lambda(t)a(t,0)\pi(t)dt \\
&= \int_0^\delta \int_0^T \lambda(t)a(t,0)\pi(t)\frac{\partial}{\partial x}h(t,x)dtdx.
\end{aligned}
\tag{A.19}
$$

By substituting (A.17)- (A.19) into equation (A.16), we obtain

$$
\begin{aligned}
0 = &-\int_0^\delta \int_0^T \frac{\partial}{\partial x}h(t,x)\left[\frac{\partial}{\partial t}\eta(t,x) - q(t,x)\frac{\partial}{\partial x}\eta(t,x) + \int_{0+}^x \lambda a(t,z)\eta(t,dz)\right. \\
&\qquad\qquad \left. + \lambda(t)a(t,x)\pi(t)\right]dtdx \\
&-\int_\delta^\infty \int_0^T \frac{\partial}{\partial x}h(t,x)\left[\frac{\partial}{\partial t}\eta(t,x) - q(t,x)\frac{\partial}{\partial x}\eta(t,x) + \int_{x-\delta}^x \lambda a(t,z)\eta(t,dz)\right]dtdx,
\end{aligned}
\tag{A.20}
$$

which is true for any test function $h(t,x) \in C_c^\infty((0,T)\times\mathbb{R}_+)$. According to the first term of the right hand side of (A.20), we have

$$0 = \frac{\partial}{\partial t}\eta(t,x) - q(t,x)\frac{\partial}{\partial x}\eta(t,x) + \int_{0+}^{x}\lambda(t)a(t,z)\eta(t,dz) + \lambda(t)a(t,x)\pi(t), \quad 0 < x < \delta. \tag{A.21}$$

According to the second term of the right hand side of (A.20), we have

$$0 = \frac{\partial}{\partial t}\eta(t,x) - q(t,x)\frac{\partial}{\partial x}\eta(t,x) + \int_{x-\delta}^{x}\lambda(t)a(t,z)\eta(t,dz), \quad x > \delta. \tag{A.22}$$

Since X_t has point mass accumulated at $x = 0$ so that $\eta(t,0) \neq \lim_{x\to 0+}\eta(t,x)$, we determine the boundary probability $\pi(t)$ by the relation

$$\pi(t) = \lim_{x\to 0+}(1-\eta(t,x)) = 1 - \eta(t,0+). \tag{A.23}$$

The three pieces of equations (A.21) - (A.23) constitute the transport equation of reserves distribution given in Proposition 3.1.

□

A.4 Proof of Proposition 4.2

About the mean field game equilibrium in fluid limit, we need to understand the partial differential equations associated with the mean field game model in fluid limit, which are summarized in the following lemmas A.1 and A.2.

Lemma A.1. *The limiting game value function* v_0 *and reserves distribution function* $(\tilde{\pi}_0, \tilde{\eta}_0)$ *satisfy the following system of mean field game equations* (A.24) - (A.25).

$$r\tilde{v}_0(x) = \left[\left(\tilde{p}_0 - \tilde{v}_0'(x)\right)\tilde{q}_0^*(x) - C_q(\tilde{q}_0^*(x))\right] + \left[-C_a(\tilde{a}_0^*(x)) + \tilde{a}_0^*(x)\lambda\delta\tilde{v}_0'(x)\right], 0 \le x, \tag{A.24}$$

$$\begin{cases} 0 = -\lambda\delta\tilde{a}_0^*(0)\tilde{\pi}_0 - \tilde{q}_0^*(0+)\tilde{\eta}_0'(0+), \\ 0 = \left(-\lambda\delta\tilde{a}_0^*(x) + \tilde{q}_0^*(x)\right)\tilde{\eta}_0'(x), \quad x > 0, \end{cases} \tag{A.25}$$

where the optimal production rate $\tilde{q}_0^*$ *and exploration rate* $\tilde{a}_0^*$ *are given by*

$$\begin{aligned} \tilde{q}_0^*(x) &= \frac{1}{\beta_1}\left(L - \tilde{Q}_0 - \kappa_1 - \tilde{v}_0'(x)\right)^+, \\ \tilde{a}_0^*(x) &= \frac{1}{\beta_2}\left(\lambda\delta\tilde{v}_0'(x) - \kappa_2\right)^+, \end{aligned} \tag{A.26}$$

with $\tilde{Q}_0$ *uniquely determined by the equation*

$$\tilde{Q}_0 = -\int_0^\infty \frac{1}{\beta_1}\left(L - \kappa_1 - \tilde{v}_0'(x) - \tilde{Q}_0\right)^+ \tilde{\eta}_0(dx),$$

and the equilibrium price is

$$\tilde{p}_0 = L + \int_0^\infty \tilde{q}_0^*(z)\tilde{\eta}_0(dz). \tag{A.27}$$

Proof. To obtain the HJB equation (A.24) of limiting game value function $\tilde{v}_0(x)$ and the associated optimal production controls (A.26), we let $\epsilon \to 0$, i.e. $\delta\epsilon \downarrow 0$

and $\lambda/\epsilon \uparrow \infty$ in the HJB equation (3.8) and the associated optimal production and exploration controls (3.9)-(3.10)

$$\begin{aligned}
\tilde{a}_0^*(x) = \lim_{\epsilon\to 0}\tilde{a}_\epsilon^*(x) &= \lim_{\epsilon\to 0}\frac{1}{\beta_2}\left[\lambda_\epsilon\left(\tilde{v}_\epsilon(x+\delta_\epsilon)-\tilde{v}_\epsilon(x)\right)-\kappa_2\right]^+\\
&= \lim_{\epsilon\to 0}\frac{1}{\beta_2}\left[\frac{\lambda}{\epsilon}\left(\tilde{v}_\epsilon(x+\delta\epsilon)-\tilde{v}_\epsilon(x)\right)-\kappa_2\right]^+\\
&= \frac{1}{\beta_2}\left(\lambda\delta\tilde{v}_0'(x)-\kappa_2\right)^+,\\
\tilde{q}_0^*(x) = \lim_{\epsilon\to 0}\tilde{q}_\epsilon^*(x) &= \lim_{\epsilon\to 0}\frac{1}{\beta_1}\left(L-\tilde{Q}_\epsilon-\kappa_1-\tilde{v}_\epsilon'(x)\right)^+\\
&= \frac{1}{\beta_1}\left(L-\tilde{Q}_0-\kappa_1-\tilde{v}_0'(x)\right)^+,
\end{aligned}$$

where

$$\begin{aligned}
\tilde{Q}_0 = \lim_{\epsilon\to 0}\tilde{Q}_\epsilon &= \lim_{\epsilon\to 0} -\int_0^\infty \frac{1}{\beta_1}\left(L-\kappa_1-\frac{\partial}{\partial x}\tilde{v}_\epsilon(x)-\tilde{Q}_\epsilon\right)^+\tilde{\eta}_\epsilon(dx)\\
&= -\int_0^\infty \frac{1}{\beta_1}\left(L-\kappa_1-\frac{\partial}{\partial x}\tilde{v}_0(x)-\tilde{Q}_0\right)^+\tilde{\eta}_0(dx).
\end{aligned}$$

Similarly, we send $\epsilon \downarrow 0$ to obtain the fluid limit transport equation (A.25). Note that as $\epsilon \downarrow 0$ the integral term $\int_{x-\delta_\epsilon}^x \lambda_\epsilon\tilde{a}_\epsilon(z)\tilde{\eta}_\epsilon(dz)$ in the third case $\delta_\epsilon < x$ of (3.21c) converges to

$$\lim_{\epsilon\to 0}\int_{x-\delta_\epsilon}^x \lambda_\epsilon\tilde{a}_\epsilon(z)\tilde{\eta}_\epsilon(dz) = \lim_{\epsilon\to 0}\int_{x-\delta\epsilon}^x \frac{\lambda}{\epsilon}\tilde{a}_\epsilon(z)\tilde{\eta}_\epsilon(dz) = \lambda\tilde{a}_0(x)\frac{\partial}{\partial x}\tilde{\eta}_0(x).$$

□

Lemma A.2 states production and exploration strategies with no reserves $x = 0$.

Lemma A.2 (Equilibrium production and exploration of fluid limit on boundary $x = 0$)**.** *The equilibrium production and exploration rates in fluid limit on the boundary* $x = 0$ *satisfy* (4.15)

Proof. On the boundary $x = 0$ where there is no reserves, we must have $\delta\lambda\tilde{a}_0^*(0) \geq \tilde{q}_0^*(0) \geq 0$, i.e., the rate of reserves additions must be greater than or equal to production rate. If $a_0^*(0) = 0$, it follows directly that $q_0^*(0) = \lambda\delta a_0^*(0) = 0$.

Now we consider the case that $a_0^*(0) > 0$. Since $\tilde{q}_0^*(x) \geq 0$ is increasing and $\tilde{a}_0^*(x)$ is decreasing to 0 as x increases, we must have some point $x^* \geq 0$ such that $q_0^*(x^*) = \lambda\delta a_0^*(x^*)$. Note that once reserves process X_t reaches the level x^*, it will remain unchanged since production rate $\tilde{q}_0^*(x^*)$ is balanced by the rate of reserves increment $\lambda\delta\tilde{v}_0^*(x^*)$ at $X_t = x^*$.

Suppose by contradiction that $x^* > 0$, then we have

$$\begin{aligned}
\tilde{v}_0(x^*) &= \int_0^\infty \left[\tilde{p}_0\tilde{q}_0^*(X_t) - C_q(\tilde{q}_0^*(X_t)) - C_a(\tilde{a}_0^*(X_t))\right] e^{-rt}dt \,\Big|_{X_0=x^*} \\
&= \int_0^\infty \left[\tilde{p}_0\tilde{q}_0^*(x^*) - C_q(\tilde{q}_0^*(x^*)) - C_a(\tilde{a}_0^*(x^*))\right] e^{-rt}dt \,\Big|_{X_0=x^*} \\
&= \int_0^\infty \left[\tilde{p}_0\tilde{q}_0^*(x^*) - C_q(\tilde{q}_0^*(x^*)) - C_a(\tilde{a}_0^*(x^*))\right] e^{-rt}dt \,\Big|_{X_0=0} \leq \tilde{v}_0(0),
\end{aligned}$$

where the third equality is due to $q_0^*(x^*) = \lambda\delta a_0^*(x^*)$ so that the initial reserves level X_0 does not influence the game value, and the last inequality is due to that the constant production rate $\tilde{q}_0^*(x^*)$ and exploration rate $\tilde{a}_0^*(x^*)$ are not necessarily optimal at $X_t = 0$.

Letting $\tau := \inf\left\{t \geq 0 : \int_0^t q_0^*(0)ds = x^*\right\}$ with $\tau = \infty$ if $\int_0^t q_0^*(0)ds < x^*$ for all $t \geq 0$, we have

$$\begin{aligned}
\tilde{v}_0(x^*) &\geq \int_0^\tau \left[\tilde{p}_0 q_0^*(0) - C_q(q_0^*(0))\right] \mathrm{e}^{-rt} dt + \mathrm{e}^{-r\tau} \tilde{v}_0(0) \Big|_{X_0 = x^*} \\
&> \int_0^\tau \left[\tilde{p}_0 q_0^*(0) - C_q(q_0^*(0)) - C_a(a_0^*(0))\right] \mathrm{e}^{-rt} dt + \mathrm{e}^{-r\tau} \tilde{v}_0(0) \Big|_{X_0 = x^*} \\
&= \int_0^\tau \left[\tilde{p}_0 q_0^*(0) - C_q(q_0^*(0)) - C_a(a_0^*(0))\right] \mathrm{e}^{-rt} dt + \mathrm{e}^{-r\tau} \tilde{v}_0(0) \Big|_{X_0 = 0} \\
&= \tilde{v}_0(0),
\end{aligned}$$

where the strict inequality " $>$ " is due to the assumption that $a_0^*(0) > 0$.

The above two inequalities $\tilde{v}_0(0) \geq \tilde{v}_0(x^*)$ and $\tilde{v}_0(x^*) > \tilde{v}_0(0)$ contradict each other. Then we have that $x^* = 0$, and thus $\tilde{q}_0^*(0) = \lambda\delta\tilde{a}_0^*(0)$.

□

With the Lemmas A.1 and A.2, we are ready to prove Proposition 4.2.

Proof of proposition 4.2. (i). Since $\tilde{q}_0^*(x) > 0$ and $\tilde{a}_0^*(x) = 0$ for $x > 0$, we have $\tilde{\eta}_0'(x) = 0$ for $x > 0$, according to the equation (A.25) in the interior $x > 0$. Since there is not probability density for $x > 0$, we have boundary probability equal to 1, i.e. $\tilde{\pi}_0 = 1$.

(ii). According to the conclusion (i) that $\tilde{\pi}_0 = 1$, we can determine $\tilde{Q}_0$ by

$$\begin{aligned}
\tilde{Q}_0 &= -\int_0^\infty \frac{1}{\beta_1} \left(L - \kappa_1 - \tilde{v}_0'(x) - \tilde{Q}_0\right)^+ \tilde{\eta}_0(dx) \\
&= \frac{1}{\beta_1} \left(L - \kappa_1 - \tilde{v}_0'(0) - \tilde{Q}_0\right)^+,
\end{aligned}$$

which gives $\tilde{Q}_0 = \frac{1}{1+\beta_1}\left(L-\kappa_1-\tilde{v}_0'(0)\right)^+$. According to (A.26) and the result $\tilde{Q}_0 = \frac{1}{1+\beta_1}\left(L-\kappa_1-\tilde{v}_0'(0)\right)^+$, the equilibrium production rate at $x=0$ is

$$\begin{aligned}
\tilde{q}_0^*(0) &= \frac{1}{\beta_1}\left(L-\kappa_1-\tilde{v}_0'(0)-\tilde{Q}_0\right)^+ \\
&= \frac{1}{\beta_1}\left[L-\kappa_1-\tilde{v}_0'(0)-\frac{1}{1+\beta_1}\left(L-\kappa_1-\tilde{v}_0'(0)\right)^+\right]^+ \\
&= \frac{1}{1+\beta_1}\left(L-\kappa_1-\tilde{v}_0'(0)\right)^+ . \qquad \text{(A.28)}
\end{aligned}$$

By substituting production rate (A.28) and exploration effort (A.26) with $x = 0$ into the equation $\tilde{q}_0^*(0) = \lambda\delta\tilde{a}_0^*(0)$ and solving for $\tilde{v}_0'(0)$, we obtain

$$\tilde{v}_0'(0) = \frac{(L-\kappa_1)\beta_2+\kappa_1(1+\beta_1)}{\beta_2+(1+\beta_1)\lambda\delta}. \qquad \text{(A.29)}$$

Then by substituting the above $\tilde{v}_0'(0)$ into (A.28) we have

$$\tilde{q}_0^*(0) = \frac{[(L-\kappa_1)\lambda\delta-\kappa_2]^+}{\beta_2+(1+\beta_1)\lambda\delta}.$$

The above $\tilde{q}_0^*(0)$ together with the conclusion (i) gives equilibrium total production

$$\tilde{Q}_0 = -\int_0^\infty \tilde{q}_0^*(x)\tilde{\eta}_0(dx) = \tilde{\pi}_0\tilde{q}_0^*(0) = \frac{[(L-\kappa_1)\lambda\delta-\kappa_2]^+}{\beta_2+(1+\beta_1)\lambda\delta}.$$

□

A.5 Proof of Lemma 4.1

Proof. Similar to Lemma A.2, at $x = 0$ we have

$$q_0^*(t,0) = \lambda\delta a_0^*(t,0). \tag{A.30}$$

By substituting (4.10)-(4.11) into (A.30), we obtain the boundary condition $\frac{\partial}{\partial x}v_0(t,0)$,

$$\frac{\partial}{\partial x}v_0(t,0) = \frac{\beta_2(p_0(t)-\kappa_1)+\beta_1\lambda\delta\kappa_2}{\beta_1\lambda^2\delta^2+\beta_2}.$$

Substituting (4.12) into (4.10)-(4.11), we obtain $a_0^*(t,0)$ and $q_0^*(t,0)$ in explicit form

$$a_0^*(t,0) = \frac{\lambda\delta(p_0(t)-\kappa_1)-\kappa_2}{\beta_1\lambda^2\delta^2+\beta_2}, \tag{A.31}$$

$$q_0^*(t,0) = \frac{\lambda^2\delta^2(p_0(t)-\kappa_1)-\lambda\delta\kappa_2}{\beta_1\lambda^2\delta^2+\beta_2}. \tag{A.32}$$

By substituting (4.12), (A.31), and (A.32) into the HJB equation (4.8), we obtain the following linear first-order differential equation for $v_0(\cdot,0)$:

$$0 = \frac{\partial}{\partial t}v_0(t,0) - rv_0(t,0) + \frac{1}{2}\left[(a_0^*(t,0))^2 + (q_0^*(t,0))^2\right], \qquad 0 < x,\ 0 \le t < T,$$

which admits an explicit solution

$$v_0(t,0) = v_0(T,0)\mathrm{e}^{-r(T-t)} + \int_t^T \frac{1}{2}\left[(a_0^*(s,0))^2 + (q_0^*(s,0))^2\right]\mathrm{e}^{-r(s-t)}ds$$

that matches (4.13) since $v_0(T,0) = 0$. □

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